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SPECIMEN DISQUISITIONIS
CURVARUM, QUÆ IN IIS QUARTI ORDINIS ÆQUATIONIBUS CONTINENTUR,
IN QUIBUS QUANTITATIBUS VARIABILIBUS
X, Y PARES TANTUM EXPONENTES TRIBUUNTUR.

DISSERTATIO

QUAM

AMPLISSIMI PHILOSOPHORUM ORDINIS AUCTORITATE

IN

ACADEMIA VRATISLAVIENSI

PRO

SUMMIS IN PHILOSOPHIA HONORIBUS RITE OBTINENDIS

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AUCTOR

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VRATISLAVIÆ,

TYPIS HENRICI RICHTERI.

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DOCTISSIMO OPTIMEQUE DE SE MERITO

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TANQUAM GRATI ANIMI TESTEM SACRUM ESSE VOLUIT

AUCTOR.

Aequatio generalis curvarum disquirendarum est haec:

I. $y^4\cdot(\alpha-\beta)+y^2x^2\cdot(\gamma-\delta)+y^2\cdot(\varepsilon-\xi)+(\eta-\vartheta)+x^2\cdot(\varkappa-\lambda)+x^4\cdot(\mu-\nu)=o$
in qua aequatione quantitates α, β, γ, δ, ε, ζ, η, ϑ, $\varkappa$, λ, μ, ν quemlibet valorem obtinere possunt.

Aequatio data transit, si $(\alpha-\beta)$ per A designatur, $(\gamma-\delta)$ per B, $(\varepsilon-\zeta)$ per C, $(\eta-\vartheta)$ per D, $(\varkappa-\lambda)$ per E, $(\mu-\nu)$ per F, in iis casibus, quibus $\alpha>\beta$, $\mu>\nu$ aut $\alpha<\beta$, $\mu<\nu$ est, in aequationem hanc.

II. $$Ay^4\underset{0}{+}Bx^2y^2\underset{0}{+}Cy^2\underset{0}{+}D\underset{0}{+}Ex^2+Fx^4=o;$$

in iis, quibus $\alpha>\beta$, $\mu<\nu$ aut $\alpha<\beta$, $\mu>\nu$ sumitur, in aequ.

III. $$Ay^4\underset{0}{+}Bx^2y^2\underset{0}{+}Cy^2\underset{0}{+}D\underset{0}{+}Ex^2-Fx^4=o;$$

in iis, quibus $\alpha=\beta$, $\mu\gtrless\nu$ est, in aequ.

IV. $$Fx^4\underset{0}{+}Bx^2y^2\underset{0}{+}Ex^2\underset{0}{+}D\underset{0}{+}Cy=o;$$

in iis, quibus $\alpha\gtrless\beta$, $\mu=\nu$ ponitur, in aequ.

V. $$Ay^4\underset{0}{+}Bx^2y^2\underset{0}{+}Cy^2\underset{0}{+}D\underset{0}{+}Ex^2=o;$$

in iis denique, quibus $\alpha=\beta$, $\mu=\nu$ est, in aequ.

VI. $$Bx^2y^2\underset{0}{+}Ex^2\underset{0}{+}D\underset{0}{+}Cy^2=o;$$

Cum aequ. V. si $\left.\begin{aligned}x&=x_1\ \cos.\ \psi+y_1\ sin.\ \psi\\ y&=y_1\ \cos.\ \psi=x_1\ sin.\ \psi\end{aligned}\right\}$ et $\psi=90^0$ substituitur
cum aequ. IV. conveniat, nobis disquirendae sunt aequationes II. III. IV. VI.

De curvis

aequatione hac: $Ay^4 + Bx^2y^2 + Cy^2 + D + Ex^2 + Fx^4 = o$ expressis.

§. 2. Aequatio data, in qua quantitates B, C, D, E ad libitum ponendae sunt sive respicis signum sive valorem; F vero et A quantitates semper positivae sunt, scribi potest.

1) $$y^4 + \frac{B}{A}x^2y^2 + \frac{C}{A}y^2 + \frac{D}{A} + \frac{E}{A}x^2 + \frac{F}{A}x^4 = o;$$

2) $$\text{aut } y^4\frac{A}{F} + x^2y^2\frac{B}{F} + y^2\frac{C}{F} + \frac{D}{F} + x^2\frac{E}{F} + x^4 = o;$$

Aequatio 2 commutabitur, si $y = z\ cos.\ \psi - u\ sin.\ \psi$, $\psi = 90^0$ ponitur.
$x = z\ sin.\ \psi + u\ cos.\ \psi$

et $\frac{A}{F} = e$, $\frac{B}{F} = b$, $\frac{C}{F} = d$, $\frac{D}{F} = c$, $\frac{E}{F} = a$, in aequationem hanc:

3) $$z^4 + bz^2u^2 + z^2a + c + du^2 + eu^4 = o;$$

aequ. 1) si $y = z$, $x = u$, $\frac{B}{A} = b$, $\frac{C}{A} = a$, $\frac{D}{A} = c$, $\frac{E}{A} = d$, $\frac{F}{A} = e$ sumitur, in aequ. hanc:

4) $z^4 + bz^2u^2 + az^2 + c + du^2 + eu^4 = o$, quae aequatio prorsus eadem est, atque aequ. 3). Ergo tibi licebit ponere, ut sit in aequ. 1 $F \leqq A$, et quia $\frac{F}{A} = e$ est, $e \leqq 1$.

Aequatio 1 soluta praebet $y = \sqrt{-\frac{b}{2}\left(x^2 + \frac{a}{b}\right) + \sqrt{\frac{b^2}{4}\left(x^2 + \frac{a}{b}\right)^2 - e\left(x^2 + \frac{d}{e}x^2 + \frac{c}{e}\right)}}$.

Quod, sive x positivum est, sive negativum, valor quantitatis y non variatur et in universum y quatuor valores obtinet, quorum bini aequales quidem, sed signis oppositis praediti sunt, apparet, omnes curvas aequ. data expressas axibus et abscissarum et ordinatarum in duas partes symmetricas et aequales dividi. Quamobrem sufficiet, de illis curvarum punctis, quae ad positivas coordinatas pertinent, accuratius agere.

Fieri potest, ut vel omnes valores, quos y offert, simul existant, vel omnes aut duo imaginarii sint. Quorum utrum ubi occurrat, nunc investigandum est in

Cap. I. de valoribus quantitatis y et realibus et imaginariis.

§. 3. Apertum est, quantitatem y realem esse, si trinomium $e\left(x^4 + \frac{d}{e}x^2 + \frac{c}{e}\right) = e\varphi(x)$ negativa quantitas est, imaginariam contra, si negativum est polynomium $\frac{b^2}{4}\left(x^2 + \frac{a}{b}\right)^2 - e\left(x^4 + \frac{d}{e}x^2 + \frac{c}{e}\right)$.

Evanescente functione $\varphi(x)$ fit

$$y = o \text{ aut } = \sqrt{-b\left(x^2 + \frac{a}{b}\right)} \text{ si est } a+,\ b+,$$

$$y = o \text{ aut } = \sqrt{b\left(x^2 + \frac{a}{b}\right)} \text{ si est } a-,\ b-,$$

$$y=o \text{ aut } =\sqrt{\;b\left\{x^2-\frac{a}{b}\right\}} \text{ si est } a+,\ b-,$$

$$y=o \text{ aut } =\sqrt{-b\left\{x^2-\frac{a}{b}\right\}} \text{ si est } a-,\ b+,$$

$$y=o \text{ aut } =\sqrt{\mp bx^2} \qquad \text{ si est } a=o,\ b\pm,$$

$$y=o \text{ aut } =\sqrt{\mp a} \qquad \text{ si est } b=o,\ a\pm,$$

$$y=o \text{ si est } a=b=o$$

Si quilibet valor quantitatis x in aequ. 1 §. 2 substitutus efficit, ut fiant quantitates positivae et $\varphi(x)=\left\{x^4+\frac{d}{e}x^2+\frac{c}{e}\right\}$ et polynomium $\frac{b^2}{4}\left\{x^2\pm\frac{a}{b}\right\}^2-e\left\{x^4+\frac{d}{e}x^2+\frac{c}{e}\right\}$.

y obtinet valores imaginarios duos in eo casu, quo $a+$, $b+$, est,

valores reales duos in eo, quo $a-$, $b-$,

valores aut reales aut imaginarios in iis, quibus $\left\{\begin{matrix} a+,\ b-, \\ a-,\ b+, \end{matrix}\right\}$ est, prout $\mp\frac{b}{2}\left\{x^2-\frac{a}{b}\right\}$ est quantitas aut positiva aut negativa.

Evanescente polynomio $\frac{b^2}{4}\left\{x^2+\frac{a}{b}\right\}^2-e\left\{x^4+\frac{d}{e}x^2+\frac{c}{e}\right\}$ fit.

$$y=\sqrt{-\frac{b}{2}\left\{x^2+\frac{a}{b}\right\}} \text{ si est } a+,\ b+$$

$$y=\sqrt{\frac{b}{2}\left\{x^2+\frac{a}{b}\right\}} \text{ si est } a-,\ b-$$

$$y=\sqrt{\frac{b}{2}\left\{x^2-\frac{a}{b}\right\}} \text{ si est } a+,\ b-$$

$$y=\sqrt{-\frac{b}{2}\left\{x^2-\frac{a}{b}\right\}} \text{ si est } a-,\ b+$$

$$y=\sqrt{\mp\frac{b}{2}x^2} \qquad \text{ si est } a=o,\ b\pm$$

$$y=\sqrt{\mp\frac{a}{2}} \qquad \text{ si est } a\pm,\ b=o.$$

Quum difficillime dijudicandum sit, utrum $\mp\frac{b}{2}\left\{x^2-\frac{a}{b}\right\}$ quantitas positiva sit, an negativa, nisi inter quantitates constantes a, b, c, d, e finitam certamque rationem constituas, aptum esse videtur, formulam hanc:

$$y=\sqrt{\pm\frac{b}{2}\left\{x^2-\frac{a}{b}\right\}\pm\sqrt{\frac{b^2}{4}\left\{x^2-\frac{a}{b}\right\}^2-e\left\{x^4+\frac{d}{e}x^2+\frac{c}{e}\right\}}}$$

in aliam ad calculum magis accommodatam hoc modo commutare.

§. 4. Quod fractiones $\frac{d}{e}, \frac{c}{e}$ quemlibet valorem finitum aut positivum aut negativum adoptare vel etiam$=o$ esse possunt, libebit etiam iis tales valores tribuere, ut trinomia

$$\frac{b^2}{4}\left\{x^4-\frac{2a}{b}x^2+\frac{a^2}{b^2}\right\}$$

$$\text{et } e\left\{x^4+\frac{d}{e}x^2+\frac{c}{e}\right\}$$

eodem quantitatis x valore substituto evanescant. Quod cum, ut in aequationum doctrina demonstratur, fieri non possit, nisi ponatur $\frac{d}{e}=-\frac{2a}{b}, \frac{c}{e}=\frac{a^2}{b^2}$, hos valores in locum $\frac{d}{e}, \frac{c}{e}$ substitue, quo facto prodit

$$y=\sqrt{\left\{x^2-\frac{a}{b}\right\}\left\{\pm\frac{b}{2}\pm\sqrt{\left\{\frac{b^2}{4}-e\right\}}\right\}}$$

Cum vero necesse sit, ut trimonium $e\left\{x^4+\frac{d}{e}x^2+\frac{c}{e}\right\}$ complectatur hoc. $e\left\{x^4-\frac{2a}{b}x^2+\frac{a^2}{b^2}\right\}$ ponendum est $\frac{d}{e}=-\frac{2a}{b}(1+p)=-2g^2(1+p)$,

$$\frac{c}{e}=\frac{a^2}{b^2}(1+n)=g^4\cdot(1+n),$$

quibus in formulis p et n pari modo ut d et c in finibus valorum $+\infty$ et $-\infty$ inclusa sunt, quantitatibus a et b valores finiti tribuendi.

Substitutis valoribus $\left\{\begin{array}{l}\frac{d}{e}=-\frac{2a}{b}\cdot(1+p)=-2g^2\cdot(1+p)\\ \frac{c}{e}=\frac{a^2}{b^2}\cdot(1+n)=\quad g^4\cdot(1+n)\end{array}\right\}$ in aequ. 1 §. 2 fit.

$$5)\quad y^*=\sqrt{-\frac{b}{2}\left\{x^2-g^2\right\}\pm\sqrt{\frac{b^2}{4}\left\{x^2-g^2\right\}^2-e\left\{x^4-2g^2x^2\left\{1+p\right\}+g^4\left\{1+n\right\}\right\}}}$$

si $b+$, $a-$ est.

$$6)\quad y=\sqrt{\frac{b}{2}\left\{x^2-g^2\right\}\pm\sqrt{\frac{b^2}{4}\left\{x^2-g^2\right\}^2-e\left\{x^4-2g^2x^2\left\{1+p\right\}+g^4\left\{1+n\right\}\right\}}}$$

si est $b-$, $a+$

*) Nota. Si ponis, ut sit $e=1$, $b=2$, $p-$, $p=2$, $g=a$, $n-$, $ng^4=f^4$ aequ. 5 exprimitur illa curva, quam Cassini primus disquisivit, Brandes in libro de geometria sublimiore tom I. pag. 170 sequu. Si ponis contra, ut sit $e=1$, $b=2$, $p-$, $p=2$, $n-$, $n=1$ prodit aequ. illius curvae, quae lemniscata vocatur.

Si simili modo in eo casu, quo a et b signis iisdem praediti sunt, $\frac{d}{e}=\frac{2a}{b}(1+q)=2g^2(1+q)$, $\frac{c}{e}=\frac{a^2}{b^2}(1+n)=g^4\cdot(1+n)$ in aequ. 1 §. 2 substitueris, prout $a+$, $b+$ est, aut $b-$, $a-$, invenies

$$7)\quad y=\sqrt{-\frac{b}{2}\left\{x^2+g^2\right\}\pm\sqrt{\frac{b^2}{4}\left\{x^2+g^2\right\}^2-e\left\{x^4+2g^2x^2\left\{1+q\right\}+g^4\left\{1+n\right\}\right\}}}$$

$$8)\quad y=\sqrt{\frac{b}{2}\left\{x^2+g^2\right\}\pm\sqrt{\frac{b^2}{4}\left\{x^2+g^2\right\}^2-e\left\{x^4+2g^2x^2\left\{1+q\right\}+g^4\left\{1+n\right\}\right\}}}$$

Disquirendae igitur sunt aequationes 5. 6. 7. 8. et

$$\text{aequ. 9)}\quad y=\sqrt{-\frac{b}{2}x^2\pm\sqrt{\frac{b^2}{4}x^4-e\left\{x^4+\frac{d}{e}x^2+\frac{c}{e}\right\}}}\quad b+,\ a=o$$

$$10)\quad y=\sqrt{+\frac{b}{2}x^2\pm\sqrt{\frac{b^2}{4}x^4-e\left\{x^4+\frac{d}{e}x^2+\frac{c}{e}\right\}}}\quad b-,\ a=o$$

$$11)\quad y=\sqrt{-\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-e\left\{x^4+\frac{d}{e}x^2+\frac{c}{e}\right\}}}\quad b=o,\ a+$$

$$12)\quad y=\sqrt{\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-e\left\{x^4+\frac{d}{e}x^2+\frac{c}{e}\right\}}}\quad b=o,\ a-$$

$$13)\quad y=\sqrt[4]{-e\left\{x^4+\frac{d}{e}x^2+\frac{c}{e}\right\}}\quad b=a=o.$$

§. 5. Ut cognoscamus, quibus valoribus quantitatis x reales aut imaginarii valores quantitatis y respondeant, primum exquirendi sunt ii valores quantitatis x, qui in aequ. 5. 6, de quibus mihi nunc agendum est, substituti efficiunt, ut polynomia

$$14)\quad \left\{x^4-2g^2x^2\cdot(1+p)+g^4\cdot(1+n)\right\}=\vartheta(x)$$

$$15)\quad \frac{b^2}{4}(x^2-g^2)^2-e\left\{x^4-2g^2x^2\cdot(1+p)+g^4\cdot(1+n)\right\}=F(x)$$

evanescant; deinde limites explorandi sunt, inter quos polynomia $\vartheta(x)=o$, $F(x)$ valores negativos adoptent, inter quos positivos. Qui limites cum tum demum inveniri possint, si aequationes $\vartheta(x)=o$ $F(x)=o$ solveris et, utrum, substituto $x=+\infty$, polynomia commemorata positiva fiant, an negativa, cognoveris, necesse est, ut ex aequu. $\vartheta(x)=o$, $F(x)=o$ eae aequationes, quae $x=+\infty$ substituto positivos valores adoptent, quae negativos, deducantur, priusquam aequationes 14. 15 solvantur.

§. 6. Substituo $x=+\infty$ fit $F(x)=-\left\{e-\frac{b^2}{4}\right\}x^4=-\infty$ si est $e>\frac{b^2}{4}$

fit $F(x)=\left\{\frac{b^2}{4}-e\right\}x^4=+\infty$ si est $e<\frac{b^2}{4}$

Substituto $x = -\infty$ fit $F(x) = \frac{b^2}{2} \cdot x^2 \cdot g^2 p = +\infty$ si est $p+$ ⎫ $e = \frac{b^2}{4}$

fit $F(x) = -\frac{b^2}{2} \cdot x^2 \cdot g^2 p = -\infty$ si est $p-$ ⎭.

fit $F(x) = \frac{b^2}{4} \cdot g^4 n$ si est $e = \frac{b^2}{4}$, $p = o$, $n-$

fit $F(x) = -\frac{b^2}{4} \cdot g^4 n$ si est $e = \frac{b^2}{4}$, $p = o$, $n+$

quamobrem ex aequ. 15 aequationes hae derivandae sunt:

16) $-s\left\{x^4 - 2g^2 x^2 \cdot \frac{e}{s}\left(\frac{s}{e} + p\right) + g^4 \cdot \frac{e}{s}\left(\frac{s}{e} + n\right)\right\} = o$, si est $e > \frac{b^2}{4}$, $e - \frac{b^2}{4} = s$

17) $g^2 \frac{b^2}{4}\left(2px^2 - g^2 n\right) = o$ si est $e = \frac{b^2}{4}$, $p+$

18) $-g^2 \frac{b^2}{4}\left(2px^2 + g^2 n\right) = o$ si est $e = \frac{b^2}{4}$, $p-$

19) $s\left\{x^4 - 2g^2 x^2 \cdot \frac{e}{s}\left(\frac{s}{e} - p\right) + g^4 \cdot \frac{e}{s}\left(\frac{s}{e} - n\right)\right\} = o$, si est $e < \frac{b^2}{4}$, $\frac{b^2}{4} - e = s$.

§. 7. Radices aequ. 14. $x = k = g\sqrt{1 + p + \sqrt{p^2 + 2p - n}}$

$$x = l = g\sqrt{1 + p - \sqrt{p^2 + 2p - n}}$$

sunt I. imaginariae
- α. si est $p+$, $n+$, $p < -1 + \sqrt{1+n}$;
- β. si est $p-$, $n-$, $p > 1 - \sqrt{1-n}$, $n < 1$;
- γ. si est $p-$, $n+$;

II. reales aequales
- α. $k = l = g\sqrt{(1+p)}$ si est $p+$, $n+$, $p = -1 + \sqrt{1+n}$;
- β. $k = l = g$ si est $p = n = o$;
- γ. $k = l = g\sqrt{(1-p)}$ si est $p-$, $n-$, $p = 1 - \sqrt{1-n}$, $n < 1$;

III. Radix k aequ. $14 = l = o$ est, si est $p-$, $n-$, $p = n = 1$.

IV. Radix $k = o$, l imag. est, si est $p-$, $n-$, $n = 1$, $p > 1$

V. k reale, l imaginarium est,
- α, $k > g$, si est 1. $p+$, $n-$, $n > 1$,
  2. $p-$, $n-$, $n > 1$, $2p < n$,
- β, $k = g$, si est $p-$, $n-$, $n > 1$, $2p = n$,
- γ, $k < g$, si est $p-$, $n-$, $n > 1$, $2p > n$,

VI. k reale est, $l = o$
- α. $k > g$, si est 1. $p-$, $n-$, $n = 1$
  2. $p-$, $n-$, $n = 1$, $2p < 1$

β. $k=g$, si est $p-$, $n-$, $n=1$ $2p=1$,

γ. $k<g$, si est $p-$, $n-$, $n=1$, $2p>1$, $p<1$,

VII. Radices k et l reales inaequales sunt:

α. $k>l>g$, si est $p+$, $n+$, $p>-1+\sqrt{1+n}$, $2p<n$,

β. $k>g$, $l=g$, si est $p+$, $n+$, $2p=n$,

γ. $k>g$, $l<g$, si est 1. $p+$, $n+$, $2p>n$,

2. $p+$, $n-$, $n<1$,

3. $p-$, $n-$, $n<1$, $2p<n$,

δ. $k=g$, $l<g$, si est $p-$, $n-$, $n<1$, $2p=n$,

ε. $l<k<g$, si est $p-$, $n-$, $n<1$, $2p>n$, $p<1-\sqrt{1-n}$.

§. 8. Quum $x=+\infty$ sumptum efficiat, ut fiat

$$\vartheta(x)=x^4-2g^2\cdot x^2\cdot(1+p)+g^4\cdot(1+n)=+\infty,$$

functio $\vartheta(x)$ vel semper est positiva quantitas, si §. 7. I. radices k et l semper imaginariae sunt; vel tum demum, si $\left.\begin{matrix} x>k, & x<l \\ x \gtrless k=l \end{matrix}\right\}$ ut sint §. 7. II. VII. k et l quantitates reales.

vel etiam, si est $x>k$ in iis casibus, quibus (cf. §. 7) k reale est, aut $=o$, sive est $l=o$, sive imaginarium.

Quae cum ita se habeant, obtinet y unicum valorem realem in aequu. 5, 6, (§. 4.);

I. inter $x=k$ et $x=l$, si (cf. §. 7. VI. VII.) k et l reales quantitates sunt, aut si k reale, $l=o$ est;

II. si est $x<k$, k reale, l imag. (cf. §. 7. V.)

est $y=o$, aut imaginarium:

I. in aequ. 5. ubi est 1. $x=k$, $k>g$, (cf. §. 7. II. α; V. α; VI. α; VII, α, β, γ);
ubi est 2. $x=l$ $l>g$, (cf. §. 7. II; VII, α);

II. in aequ. 6. (§. 4.) ubi est 1. $x=k$ $k<g$, (cf. §. 7. II, γ; III; IV; V, γ; VI, γ; VII, ε);
ubi est 2. $x=l$ $l<g$, (cf. §. 7. II, γ; III; VI; VII. γ, δ ε);

est $y=o$, in aequu. 5, 6, (§. 4.),
ubi est 1. $x=k$, si est $k=g$, (cf. §. 7. II, β; V, β; VI, β; VII, δ);
ubi est 2. $x=l$, si est $l=g$, (cf. §. 7. II, β; VII, β);

est denique alter valor quantitatis $y=o$, alter realis,

I. in aequu. 5. ubi est 1. $x=k$ $k<g$ et $>o$ aut $k=o$ (cf. §. 7. II, γ; III; IV; V, γ; VI, γ; VII, ε);
ubi est 2. $x=l$ $l<g$ et $>o$ aut $l=o$ (cf. §. 7. II, γ; III; VI; VII, γ, δ, ε);

II. in aequu. 6 ubi est 1. $x=k$, si est $k>g$, (cf. §. 7. II, α; V, α; VI, α; VII, α, β, γ);
ubi est 2. $x=l$, si est $l>g$, (cf. §. 7. II, α; VII, α;

§. 9. Radices aequ. 19. (§. 6.) $s\left\{x^4 - 2g^2x^2\frac{e}{s}\left(\frac{s}{e}-p\right)+g\frac{4e}{s}\left(\frac{s}{e}-n\right)\right\}=o$, sunt:

$$x=f=g\sqrt{\frac{e}{s}\left\{\frac{s}{e}-p+\sqrt{p^2-2\frac{s}{e}p+\frac{s}{e}\cdot n}\right\}}$$

$$x=h=g\sqrt{\frac{e}{s}\left\{\frac{s}{e}-p-\sqrt{p^2-2\frac{s}{e}p+\frac{s}{e}\cdot n}\right\}}.$$

I. Utraque radix imaginaria est:

α. si est $p+, n+. p>\frac{s}{e}-\sqrt{\frac{s}{e}\left(\frac{s}{e}-n\right)}, \frac{e}{s}>n$; et §. 7. VII, γ, 1. $k>g, l<g$;

β. si est $p-, n-, p<-\frac{s}{e}+\sqrt{\frac{s}{e}\left(\frac{s}{e}+n\right)}$, et 1. §. 7. VII, γ, 3. $k>g\ l<g$;
2. §. 7. VI, α, 2. $k>g, l=o$;
3. §. 7. V, α, 2 $k>g$, l imag.

γ. si est $p+, n-$, et 1. §. 7. VII, γ, 2. $k>g, l<g$;
2. §. 7. VI, α, 1. $k>g, l=o$;
3. §. 7. V, α. 1. $k>g$, l imaginarium.

II. Radices f et h reales aequales sunt:

α. si est: $p+, n+, p=\frac{s}{e}-\sqrt{\frac{s}{e}\left(\frac{s}{e}-n\right)}, n<\frac{s}{e}, f=h=g\sqrt{\frac{e}{s}\left(\frac{s}{e}-p\right)}$
et §. 7. VII, γ, 1. $k>g, l<g$;

β. si est: $p=n=o, f=h=g$, et §. 7. II, β; $k=l=g$;

γ. si est: $p-, n-, p=-\frac{s}{e}+\sqrt{\frac{s}{e}\left(\frac{s}{e}+n\right)}; f=h=g\sqrt{\frac{e}{s}\left(\frac{s}{e}+p\right)}$
et 1. §. 7. VII, γ, 3. $k>g, l<g$;
2. §. 7. VI. α, 2. $k>g, l=o$;
3. §. 7. V, α, 2. $k>g$, l imag. est.

III. $f=h=o$ est, si est $p+, n+, p=n=\frac{s}{e}$
IV. $f=o$, h imag. est, si est $p+, n+, n=\frac{s}{e}, p>\frac{s}{e}$
} et §. 7. VII, γ, 1. $k>g, l<g$.

V. f reale, h imaginarium est,

α. $f<g$, si est $p+, n+, n>\frac{e}{s}, 2p>n$, §. 7. VII, γ, 1. $k>g, l<g$;

β. $f=g$, si est $p+, n+; n>\frac{s}{e}, 2p=n$; et §. 7. VII, β, $k>g, l=g$;

γ. $f>g$, si est $p+$, $n+$, $n>\frac{s}{e}$, $2p<n$, $p>-1+\sqrt{1+n}$, §. 7. VII, α, $k>l>g$;

δ. $f>g$, si est $p+$, $n+$, $n>\frac{s}{e}$, $p=-1+\sqrt{1+n}$, §. 7. II, α, $k=l=g\sqrt{(1+p)}$;

ε. $f>g$, si est $p+$, $n+$, $n>\frac{s}{e}$, $p<-1+\sqrt{1+n}$ | et §. 7. I, α, γ, radices

ζ. $f>g$, si est $p-$, $n+$, $n>\frac{s}{e}$, | k, l imaginariae sunt.

VI. f reale, $h=o$ est,

1) si est $p+$, $n+$, $n=\frac{s}{e}$, $p>\frac{s}{e}$ | cf. V.

2) si est $p-$, $n+$, $n=\frac{s}{e}$ |

VII. Radices f et h reales inaequales sunt:

α. $h<f<g$, si est $p+$, $n+$, $n<\frac{s}{e}$, $2p>n$ et §. 7. VII, γ, 1. $k>g$, $l<g$;

β. $h<g$, $f=g$, si est $p+$, $n+$, $n<\frac{s}{e}$, $2p=n$ et §. 7. VII, β, $k>g$, $l=g$;

γ. $h<g$, $f>g$, 1) si est $p+$, $n+$, $n<\frac{s}{e}$, $2p<n$ et §. 7. VII, α, $k>l>g$;

2) si est $p+$, $n+$, $n<\frac{s}{e}$, $2p<n$ et §. 7. II, α, $k=l=g\sqrt{(1+p)}$;

3) si est $p+$, $n+$, $n<\frac{s}{e}$, $2p<n$, |

4) si est $p-$, $n+$, $n<\frac{s}{e}$, | et §. 7. I, α, β, γ radices k et l imaginariae sunt;

5) si est $p-$, $n-$, $2p>n$, |

6) si est $p-$, $n-$, $2p>n$ et §. 7. II, γ, $k=l=g\sqrt{(1-p)}$;

7) si est $p-$, $n-$, $2p>n$ et §. 7. VII, ε, $l<k<g$;

8) si est $p-$, $n-$, $p=n=1$ et §. 7. III, $k=l=o$;

9) si est $p-$, $n-$, $p>1$, $n=1$, §. 7. IV, $k=o$, l imag.;

10) si est $p-$, $n-$, $n>1$, $2p>n$ et §. 7. V, γ, $k<g$, l imag.;

11) si est $p-$, $n-$, $n=1$, $2p>1$, $p<1$, et §. 7. VI, γ, $k<g$, $l=o$.

δ. $h=g$; $f>g$, si est $p-$, $n-$, $2p=n$, et §. 7. VII, δ, $k=g$, $l<g$;

2) et §. 7. VI, β, $k=g$, $l=o$;

3) et §. 7. V, β, $k=g$, l imag.

ε. $f>h>g$, si est $p-$, $n-$, et §. 7. VII, γ, 3. $k>g$, $l<g$;

2) et §. 7. VI, α, 2. $k>g$, $l=o$;

3) et §. 7. V. α, 2. $k>g$, l imag.

§. 10. Cum polynomia 14. 19. (cf. §. 5. 6.) $x=+\infty$ substituto valores positivos adoptent, sequitur, ut valores positivos obtineat polynomium 19:

I. semper, si §. 9. I. radices f, h imaginariae sunt;

II. $x<h$ substituto, $x>f$, si §. 9. II, VII radices f et h reales, sive aequales, sive inaequales sunt;

III. $x>f$ substituto, si est §. 9. V, f reale, h imag.;
si est §. 9. VI, f reale, $h=o$;

IV. $x>o$ substituto, si est §. 9. III, IV, $f=o$, $h=o$ aut imag.

E limitibus modo enumeratis, inter quos valores positivos obtinet polynomium 19, et ex iis, inter quos polynomium $e\{x^4-2g^2x^2\cdot(1+p)+g^4\cdot(1+n)\}$ (cf. §. 8.) itidem valores positivos adoptat, concludes, in aequ. 5

α. y unicum valorem realem accipere inter $x=l$ et $x=k$; $y=o$ esse, ubi est $x=k$

et I. $y=o$ esse, ubi est $x=l$, si est §. 9. V, β, $f=l=g$, $k>g$ } h imag. cf. fig. 1.
§. 9. V, γ, $f>g$, $k>l>g$ }

II. y reale esse, aut $=o$, ubi est $x=o$, cf. fig. 2, 3;

1) si est §. 9 I, β, 2, } $k>g$, $l=o$, radices f et h imag. sunt;
2) si est §. 9. I, γ. 2, }

3) si est §. 9. II. γ, 2, $k>g$, $l=o$, $f=h=g\sqrt{\frac{e}{s}\left(\frac{s}{e}+p\right)}$;

4) si est §. 9. VII, δ, 2, $h=k=g$, $l=o$, $f>g$;

5) si est §. 9. VII, ε, 2, $k>g$, $l=o$, $f>h>g$;

β. y unicum valorem realem adoptare inter $x=k$ et $x=l$; $y=o$ esse, ubi est $x=k$; y reale aut $=o$ esse, ubi est $x=l$; dnplicem valorem realem habere inter $x=l$ et $x=o$ et ubi est $x=o$; cf. fig. 4. 5;

1) si est §. 9. I, α, β, 1, γ, 1; $k>g$, $l<g$; radices f et h imag. sunt;

2) si est §. 9. II, γ, 1; $k>g$, $l<g$; $f=h=g$; $\sqrt{\frac{e}{s}\left(\frac{s}{e}+p\right)}$;

3) si est §. 9. VII, δ, 1; $k=g=h$, $l<g$, $f>g$;

4) si est §. 9. VII, ε. 1; $k>g$, $l<g$; $f>h>g$;

γ. y unicum valorem realem accipere inter $x=k$ et $x=l$ et ubi $x=f$ est; duplicem valorem realem inter $x=l$ et $x=f$; $y=o$ esse, ubi $x=k$; $y=o$ aut reale esse, ubi $x=l$ est;

1) si est §. 9. III, $f=h=o$ } $k>g$, $l<g$, cf. fig. 6. et 36;
2) si est §. 9. IV, $f=o$, h imag. }

3) si est §. 9. V, α, $f<g$, h imag., $k>g$, $l<g$, cf. fig, 7;

δ. y unicum valorem realem obtinere inter $x=k$ et $x=o$ et ubi $x=o$ est; $y=o$ esse, ubi est $x=k$, cf. fig. 8. 9:

1) si est §. 9. I, β, 3; $k>g$, l imag., radices f et h imag. sunt;

2) si est §. 9. II, γ, 3; $k>g$, l imag., $f=h=g\sqrt{\frac{e}{s}\left\{\frac{s}{e}+p\right\}}$;

3) si est §. 9. VII, δ, 3; $f>g$, $h=k=g$, l imag.;

4) si est §. 9. VII, ε, 3; $f>h>g$:

ε. *y duplicem valorem realem accipere, ubi est* $x=o$, *inter* $x=o$ *et* $x=k$; $y=o$ *esse, ubi est* $x=f=h=k=l=g$, cf. fig. 10:

si est §. 9. II, 2; $f=h=k=l=g$.

ζ. *y unicum valorem realem habere inter* $x=k$ *et* $x=l$, *ubi est* $x=g\sqrt{\frac{e}{s}\left\{\frac{s}{e}-p\right\}}$; *duplicem valorem realem, ubi est* $x=o$ *et inter* $x=o$ *et* $x=f$, *inter* $x=f$ *et* $x=l$; $y=o$ *esse, ubi est* $x=k$; $y=o$ *esse aut reale, ubi est* $x=l$, cf, fig. 11,

si est §. 9. II, 1, $k>g$, $l<g$, $f=h=g\sqrt{\frac{e}{s}\left\{\frac{s}{e}-p\right\}}$

η. *y unicum valorem realem adoptare inter* $x=k$ *et* $x=l$ *et ubi* $x=o$ *est*; $y=o$ *esse, ubi est* $x=k$, $x=l$,

1) si est §. 9. VI, β, $k>g$, $l=f=g$, $h=o$ } cf. fig. 12.
2) si est §. 9. VI, γ, $k>l>f>g$, $h=o$ }

ϑ. *y unicum valorem realem obtinere inter* $x=k$ *et* $x=l$, *ubi est* $x=o$, $x=f$; *duplicem inter* $x=l$ *et* $x=f$; *y reale esse aut* $=o$, *ubi est* $x=l$; $y=o$ *esse, ubi est* $x=k$,

si est §. 9. VI, α, f reale, $h=o$, $k>g$, $l<g$, cf. fig. 13.

ι. *y unicum valorem realem accipere inter* $x=k$ *et* $x=l$, *ubi est* $x=f$, $x=h$; *duplicem inter* $x=l$ *et* $x=f$, *inter* $x=o$ *et* $x=h$ *et ubi* $x=o$ *est*; $y=o$ *esse aut reale, ubi est* $x=l$; $y=o$ *esse, ubi est* $x=k$,

si est §. 9. VII, α, $k>g$, $l<g$, $h<f<g$, cf. fig. 14, 15.

$\varkappa$. *y unicum valorem realem obtinere inter* $x=k$ *et* $x=l$, *ubi est* $x=h$; *duplicem valorem inter* $x=h$ *et* $x=o$ *et ubi* $x=o$ *est*; $y=o$ *esse, ubi est* $x=k$, $x=l$:

1) si est §. 9. VII, β, $h<g$, $f=l=g$, $k>g$ } cf. fig. 16, 17.
2) si est §. 9. VII, γ, 1, $h<g$, $k>l>f>g$ }

λ. *y duplicem valorem realem habere inter* $x=o$ *et* $x=h$ *et ubi* $x=o$ *est*; *unicum valorem realem, ubi est* $x=h$; $y=o$ *esse, ubi est* $x=k=l$,

si est §. 9. VII, γ, 2, $f>g$, $h<g$, $k=l=g\sqrt{(1+p)}$, cf. fig. 18.

μ. *y duplicem valorem realem accipere inter* $x=o$ *et* $x=h$; *unicum, ubi est* $x=h$ cf. fig. 19, 20, 21,

si est §. 9. VII, γ, 3, 4, 5, $h<g$, $f>g$, radices k et l imag. sunt.

ν. *y duplicem valorem realem obtinere inter* $x=h$ *et* $x=k$, *inter* $x=k$ *et* $x=o$, *ubi est* $x=o$; *unicum, ubi est* $x=h$; *y reale esse aut*$=o$, *ubi est* $k=l=x$, cf. fig. 22, 23,

si est §. 9. VII, γ, 6, $h<g$, $f>g$, $k=l=g\sqrt{(1-p)}$.

ξ. *y duplicem valorem realem adoptare inter* $x=h$ *et* $x=k$, *inter* $x=l$ *et* $x=o$ *et ubi* $x=o$ *est; unicum inter* $x=k$ *et* $x=l$, *ubi est* $x=h$; $y=o$ *aut reale esse, ubi est* $x=k$, $x=l$ cf. fig. 24, 25,
si est §. 9. VII, γ, 7, $l<k<g$, $h<g$, $f>g$.

ο. *y duplicem valorem realem adoptare inter* $x=h$ *et* $x=o$; *unicum, ubi est* $x=h$; $y=o$ *aut reale esse, ubi est* $x=o$, cf. fig. 26,
si est §. 9. VII, γ, 8, 9, $h<g$, $f>g$, $k=o$, $l=o$ aut imag.

π. *y duplicem valorem realem accipere inter* $x=h$ *et* $x=k$; *unicum inter* $x=k$ *et* $x=l$, *ubi est* $x=h$; *y reale esse aut* $=o$, *ubi est* $x=k$, $x=l=o$, cf. fig. 27, 28.
si est §. 9. VII, γ, 11, $k<g$, $l=o$, $h<g$, $f>g$;

ρ. *y duplicem valorem realem obtinere inter* $x=h$ *et* $x=k$; *unicum inter* $x=k$ *et* $x=o$, *ubi est* $x=o$, $x=h$; $y=o$ *aut reale, ubi est* $x=k$, cf. fig. 29, 30,
si est §. 9. VII, γ, 10, $k<g$, l imag., $f>g$, $h<g$;

§. 11. Facile etiam cognosces, in aequ. 6.

α. *y duplicem valorem realem habere* $x>f$ *substituto; unicum, ubi est* $x=f$:

1) si est $f>g$, h imag. §. 9. V, ε, ζ,
2) si est §. 9. VI, ε ζ, $f>g$, $h=o$,
3) si est §. 9. VII, γ, 3, 4, 5, $f>g$, $h<g$

} radices k et l imag. sunt, cf. fig. 31, 32, 33, et 34.

β. *y duplicem valorem realem obtinere* $x>f$ *substituto; unicum, ubi est* $x=f$; $y=o$ *esse, ubi est* $x=k$, cf. fig. 35,
si est §. 9. VII, γ, 8, 9, $k=o$, $f>g$, $h<g$,

γ. *y duplicem valorem realem adoptare* $x>k$ *substituto*; $y=o$ *esse, ubi est* $x=k$, cf. fig. 37,
si est §. 9. II, β, $f=h=k=l$.

δ. *y duplicem valorem realem accipere* $x>k$ *substituto*; *unicum inter* $x=k$ *et* $x=l$; $y=o$ *esse, ubi est* $x=l$; $y=o$ *aut reale esse, ubi est* $x=k$, cf. fig. 38, 39, 40, 41, 42;

1) si est §. 9. I, α, $k>g$, $l<g$
2) si est §. 9. I, β, 1. $k>g$, $l<g$
3) si est §. 9. I, β, 2, $k>g$, $l=o$
4) si est §. 9. I, γ, 1, $k>g$, $l<g$
5) si est §. 9. I, γ, 2, $k>g$, $l=o$

} radices f et h imag. sunt,

6) si est §. 9. II, α, $k>g$, $l<g$, $f=h=g\sqrt{\frac{e}{s}\left(\frac{s}{e}-p\right)}$
7) si est §. 9. III, $k>g$, $l<g$, $f=h=o$,
8) si est §. 9. IV, $k>g$, $l<g$, $f=o$, h imag,
9) si est §. 9. V, α, $k>g$, $l<g$, $f<g$, h, imag,

10) si est § 9. V, β, $k>g$, $l=f=g$, h imag,

11) si est §. 9. VI, α, $k>g$, $l<g$, $f<g$, $h=o$,

12) si est §. 9. VI, β, $k>g$, $l=f=g$, $h=o$,

13) si est §. 9. VII, α, $k>g$, $l<g$, $h<f<g$,

14) si est §. 9. VII, β, $k>g$, $l=f=g$, $h<g$.

ε. *y duplicem valorem realem obtinere inter $x=f$ et $x=l$, inter $x=+\infty$ et $x=k$, ubi est $x=+\infty$; unicum inter $x=k$ et $x=l$; $y=o$ esse aut reale, ubi est $x=k$, $x=l$* cf. fig. 43, 44:

1) si est §. 9. V, γ, $f>g$, h imag, $k>l>g$,

2) si est §. 9. VI, γ, $f>g$, $h=o$, $k>l>g$,

3) si est §. 9. VII, γ, 1, $f>g$, $h<g$, $k>l>g$,

ζ. *y duplicem valorem realem obtinere $x>k$ substituto; unicum $x<k$ substituto; $y=o$ aut reale esse, ubi est $x=k$,* cf. fig. 45, 46:

1) si est §. 9. I, β, 3, $k>g$, l imag.
2) si est §. 9. I, γ, 3, $k>g$, l imag. } radices f, h imaginariae sunt.

η. *y duplicem valorem accipere inter $x=k$ et $x=+\infty$, inter $x=l$ et $x=h$, ubi est $x=+\infty$; unicum, ubi est $x=f$; $y=o$ aut reale esse, ubi est $x=k=l$,* cf. fig. 47, 48:

1) si est §. 9. V, δ, f reale, h imag.
2) si est §. 9. VI, δ, $f>g$, $h=o$
3) si est §. 9. VII, γ, 2, $f>g$, $h<g$ } $k=l=g\sqrt{(1+p)}$.

ϑ. *y duplicem valorem realem adoptare inter $x=k$ et $x=h$, inter $x=f$ et $x=+\infty$, ubi est $x=+\infty$, ubi est $x=f=h$; unicum, si $x<k$ est; $y=o$ esse aut reale, ubi est $x=k$,* cf. fig. 49,

si est §. 9. II, γ, 3, $f=h=g\sqrt{\frac{e}{s}\left(\frac{s}{e}+p\right)}$ $k>g$, l imag.

ι. *y duplicem valorem accipere inter $x=f$ et $x=+\infty$, inter $x=h$ et $x=k$, ubi est $x=+\infty$; unicum, ubi est $x=f$, $x=h$, inter $x=k$ et $x=o$ et ubi $x=o$ est; $y=o$ esse aut reale, ubi est $x=k$,* cf. fig. 50, 51.

si est §. 9. VII, ε, $k>g$, l imag., $f>h>g$.

$\varkappa$. *y duplicem valorem realem habere inter $x=h$ et $x=k$, inter $x=f$ et $x=+\infty$ et ubi $x=+\infty$; unicum, ubi est $x=f$, ubi est $x=h$ et inter $x=k$ et $x=l$; $y=o$ esse, ubi est $x=l$; $y=o$ esse, aut reale, ubi est $x=k$,* cf. fig. 52,

1) si est $f=h=g\sqrt{\frac{e}{s}\left(\frac{s}{e}+p\right)}$ $k>g$, $l=o$, §. 9. II, γ, 2

2) si est §. 9. II, γ, 1, $f=h=g\sqrt{\frac{e}{s}\left(\frac{s}{e}+p\right)}$, $k>g$, $l<g$,

3) si est §. 9. VII, ε, 2, $f>h>g$, $k>g$, $l=o$,

4) si est §. 9. VII, ε, 1, $f>h>g$, $k>g$, $l<g$.

λ. *y daplicem valorem realem adoptare inter* $x=f$ *et* $x=+\infty$, *ubi est* $x=+\infty$; *unicum inter* $x=k$ *et* $x=o$ *et ubi* $x=o$ *est*; $y=o$ *esse, ubi est* $x=k$, cf. fig. 53, 54,
1) si est §. 9. VII, δ, 3, $f>g$, $h=k=g$, l imag.,
2) si est §. 9. VII, γ, 10, $f>g$, $h<g$, $k<g$, l imag.

μ. *y duplicem valorem realem obtinere inter* $x=f$ *et* $x=+\infty$, *ubi est* $x=+\infty$; *uuicum, ubi est* $x=f$, *inter* $x=l$ *et* $x=k$; $y=o$ *esse, ubi est* $x=k$, $x=l$, cf. fig. 55:
1) si est §. 9. VII, δ, 2, $f>g$, $h=k=g$, $l=o$,
2) si est §. 9. VII, δ, 1, $f>g$, $h=k=g$, $l<g$,
3) si est §. 9. VII, γ. 11, $f>g$, $h<g$, $k<g$, $l=o$,
4) si est §. 9. VII, γ, 7, $f>g$, $h<g$, $l<k<g$.

ν. *y duplicem valorem realem accipere, ubi est* $x=+\infty$, *inter* $x=+\infty$ *et* $x=f$; *unicum, ubi est* $x=f$; $y=o$ *esse, ubi est* $x=k=l$, *si est* §. 9. VII, γ, 6,
$k=l=g\sqrt{(1-p)}$, $f>g$, $h<g$, cf. fig. 56.

Quamquam ad rem accurate exponendam omnino necessarium esse intelligas, ut reales et imaginarii quantitatis y valores ex aequu. 7, 8, 9, 10, 11, 12, 13 (§. 4.) et III, IV, VI (§. 1.) inventi demonstrentur, hoc tamen hoc loco missum jam facio, quod impensarum causa specimen disquisitionis tantum daturus sum. Integram tamen quaestionem ut alio loco matheseos gnaris proponam fortasse mihi continget.

§. 12. Quae huc usque de illis abscissarum punctis dicta sunt, ubi ordinatae $=o$ sunt et ubi reales valores accipiunt aut imaginarios, ad curvarum formam cognoscendam non sufficiunt; sed indagandum est praeterea, qualis sit cujusque puncti curvae tangens, utrum ordinatarum, an abscissarum axi parallela, num angulum cum utraque formet; num puncta duplicia vel multiplicia vel conjuncta existant et ubi sint; tum investigandum est, num curvae vel maxima vel minima habeant, num revertantur, num inflexionis puncta existant et ubi; num concavum aut convexum latus curvae axi abscissarum praebeant, denique utrum curvae, quae ramis infinitis porriguntur, asymptotas habeant, nec ne. Quia de tangentium directione, de punctis et duplicibus et conjunctis tantum primi coefficientis differentialis auxilio, de maximis et minimis, de inflexionis punctis secundi differentialis coefficientis auxilio judicari potest, in cap. II de primo coefficiente differentiali, in cap. III de secundo disseram. In cap. IV. denique in curvarum asymptotis determinandis versabor.

Cap. II. Disquisitio primi coefficientis differentialis.

§. 13. Aequatio $y^4+bx^2y^2+ay^2+c+dx^2+ex^4=o$ praebet;

20) $$\frac{dy}{dx}=-\frac{ex}{y}\left\{\frac{x^2+\frac{d}{2e}+\frac{b}{2e}y^2}{y^2+\frac{b}{2}\left\{x^2+\frac{a}{b}\right\}}\right\}$$

Ut cognoscamus, num curvae alicubi tangentes axi abscissarum parallelas habeant, valores illi quantitatum x, y exquirendi sunt, qui efficiunt, ut $\frac{dy}{dx}=o$ fiat, quod duplici modo fieri potest, aut evanescente fracto numero $\frac{x}{y}$, aut $\frac{x^2+\frac{d}{2e}+\frac{b}{2e}y^2}{y^2+\frac{b}{2}\left(x^2+\frac{a}{b}\right)}$.

Cum quisque numerus fractus $=o$ sit, si numerator $=o$, denominator finita quantitas est, aut si denominator $=\infty$, numerator quantitas finita est, primum scrutaberis, utrum, si $x=o$ aut $x^2+\frac{d}{2e}+\frac{b}{2e}y^2=o$ posueris, y et $y^2+\frac{b}{2}\left(x^2+\frac{a}{b}\right)$ valores finitos adoptent, an etiam evanescant; tum investigabis, qui et qualiscunque numeri fracti $\frac{x}{y}\left\{\frac{y^2+\frac{d}{2e}+\frac{b}{2e}y^2}{y^2+\frac{b}{2}\left(x^2+\frac{a}{b}\right)}\right\}$ valor sit, si $x=+\infty$ ponitur.

Numerator $x\left(x^2+\frac{d}{2e}+\frac{b}{2e}y^2\right)$ transit, si y^2 ex aequ. 5, 6, §. 4. substituitur in

21) $x\left\{x^4\left(e-\frac{b^2}{4}\right)-2g^2x^2\left(e-\frac{b^2}{4}+ep\right)+g^4\left(e-\frac{b^2}{4}+\frac{nb^2}{4}+2pe-\frac{pb^2}{2}+p^2e\right)\right\}$

quod polynomium $=o$ fit, si $x=o$ ponitur, aut si

22) $x^4\left(e-\frac{b^2}{4}\right)-2g^2x^2\left(e-\frac{b^2}{4}+ep\right)+g^4\left(e-\frac{b^2}{4}+n\frac{b^2}{4}+2pe-p\frac{b^2}{2}+p^2e\right)=o$

sumitur.

Aequatio 22 commutabitur in eo casu, quo $e<\frac{b^2}{4}$, $\frac{b^2}{4}-e=s$ ponitur, in aequationem.

23) $x^4-2x^2g^2\frac{e}{s}\left(\frac{s}{e}-p\right)+g^4\frac{e}{s}\left(\frac{s}{e}-\frac{nb^2}{4e}+2p\frac{s}{e}-p^2\right)=o$,

quae aequ. nunc solvenda est.

§. 14. Radices aequationis 23 sunt:

$$x=K=g\sqrt{\frac{e}{s}\left\{\frac{s}{e}-p+\sqrt{\frac{b^2}{4e}\left(p^2-2p\frac{s}{e}+\frac{s}{e}n\right)}\right\}}$$

$$x=L=g\sqrt{\frac{e}{s}\left\{\frac{s}{e}-p-\sqrt{\frac{b^2}{4e}\left(p^2-2p\frac{s}{e}+\frac{s}{e}n\right)}\right\}}.$$

Si substituitur $x=o$ aut $x=K$, aut $x=L$, fit polynomium $\left(x^2+\frac{d}{2e}+\frac{b}{2e}y^2\right)x=o$, sive K, L quantitates imaginariae sunt, sive reales; sed curvarum puncta tum tantum determinantur, si substitutis

radicibus realibus K, L aut $x=o$ in aequ. 5, 6, y imaginarios valores non obtinet. Facile intelligitur, utrum radices K, L quantitates reales sint, nec ne; sed ut cognoscamus, qualis sit valor quantitatis y, sive $x=K$ in aequ. 5, 6, substituitur, sive $x=L$, sive $x=o$, necesse est, ut de valoribus aequationibus illis expressis accuratius agamus.

§. 15. Est, si quantitatibus a et b signa opposita tribuuntur,

$$x\left\{x^2+\frac{d}{2e}+\frac{b^2}{2e}y^2\right\}=x\left\{x^2-g^2(1+p)+\frac{b}{2e}y^2\right\}=$$

$$x\left\{x^4\left(e-\frac{b^2}{4}\right)-2g^2\cdot x^2\cdot\left(e-\frac{b^2}{4}+ep\right)+g^4\cdot\left(e-\frac{b^4}{4}-\frac{nb^2}{4}+2pe-p\frac{b^2}{2}+ep^2\right)\right\}=o,$$

qua ex aequ. sequitur, ut sit in aequ. 5, 6

$$\sqrt{F(x)}=\sqrt{x^4\left(\frac{b^2}{4}-e\right)-2g^2x^2\left(\frac{b^2}{4}-e-ep\right)+g^4\left(\frac{b^2}{4}-e-en\right)}=$$

$$g^2\sqrt{e\left\{p^2+2p\left(\frac{e-\frac{b^2}{4}}{e}\right)-n\left(\frac{e-\frac{b^2}{4}}{e}\right)\right\}}=g^2\sqrt{e\cdot f(p\cdot n)}.$$

sive $x=K$, $x=L$ ponitur.

Substitutis $x=K$ et $\sqrt{F(x)}=g^2\sqrt{e\cdot f(p,n)}$ in aequ. 5, §. 4.

fit igitur $y=Y_1=\sqrt{-\frac{b}{2}\left(K^2-g^2\right)+g^2\sqrt{e\cdot f(p,n)}}$

$$y=Y_2=\sqrt{-\frac{b}{2}\left(K^2-g^2\right)-g^2\sqrt{e\cdot f(p,n)}};$$

substitutis $x=L$ et $F(x)=g^2\sqrt{e\cdot f(p,n)}$

$$y=Y_1=\sqrt{-\frac{b}{2}\left(L^2-g^2\right)+g^2\sqrt{e\cdot f(p,n)}}$$

$$y=Y_2=\sqrt{-\frac{b}{2}\left(L^2-g^2\right)-g^2\sqrt{e\cdot f(p,n)}}.$$

Pari modo ex aequ. 6 derivantur valores

$$y=Y_1=\sqrt{\frac{b}{2}\left(K^2-g^2\right)+g^2\sqrt{e\cdot f(p,n)}}$$

$$y=Y_2=\sqrt{\frac{b}{2}\left(K^2-g^2\right)-g^2\sqrt{e\cdot f(p,n)}}$$

$$y=Y_1=\sqrt{\frac{b}{2}\left(L^2-g^2\right)+g^2\sqrt{e\cdot f(p,n)}}$$

$$y = Y_2 = \sqrt{\frac{b}{2}\left\{L^2 - g^2\right\} - g^2\sqrt{e \cdot f(p,n)}}.$$

Cum b positive, $\frac{b^2}{4} > e$ sumptis et $x = K$, $y = Y_1 = \sqrt{-\frac{b}{2}\left\{K^2 - g^2\right\} + g^2\sqrt{e \cdot f(p,n)}}$

$$x = L,\ y = Y_2 = \sqrt{-\frac{b}{2}\left\{L^2 - g^2\right\} - g^2\sqrt{e \cdot f(p,n)}}$$

substitutis, polynomium $x^2 - g \cdot (1+p) + \frac{b}{2e}y^2 = o$ fiat,

cum porro, si b negative ponitur, $\frac{b^2}{4} > e$ et $x = L$, $y = Y_1 = \sqrt{\frac{b}{2}\left\{L^2 - g^2\right\} + g^2\sqrt{e \cdot f(p,n)}}$

$$x = K,\ y = Y_2 = \sqrt{\frac{b}{2}\left\{K^2 - g^2\right\} - g^2\sqrt{e \cdot f(p,n)}}$$

polynomium modo memoratum evanescat, nobis disquirendum est, num in priori casu K, L eodem tempore realia sint, quo Y_1, Y_2; in altero casu K, L eodem tempore, quo Y_2, Y_1.

§. 16. Eodem tempore reales quantitates sunt

I. radix K aequ. 23, (§. 13.) $= g\sqrt{\frac{e}{s}\left\{\frac{s}{e} - p + \sqrt{\frac{b^2}{4e}\left\{p^2 - 2p\frac{s}{e} + n\frac{s}{e}\right\}}\right\}}$

et $y = Y_1 = \sqrt{-\frac{b}{2}(K^2 - g^2) + g^2\sqrt{e \cdot f(p,n)}} = g\sqrt{\frac{e}{s}\left\{\frac{b}{2}p - \sqrt{e\left\{p^2 - 2p\frac{s}{e} + n\frac{s}{e}\right\}}\right\}}$

1) si est $p+$, $n-$, $p > \frac{s}{e} + \sqrt{\frac{b^2}{4e}\left\{\frac{s}{e} + n\right\}}$;

2) si est $p+$, $n+$, $p > -1 + \sqrt{1+n}$, $n > \frac{s}{e}$;

3) si est $p+$, $n+$, $p > \frac{s}{e} + \sqrt{\frac{b^2}{4e}\left\{\frac{s}{e} - n\right\}}$, $\frac{s}{e} \geqq n$;

4) si est $p+$, $n+$, $p < \frac{s}{e} - \sqrt{\frac{s}{e}\left\{\frac{s}{e} - n\right\}}$ $\frac{s}{e} \geqq n$, $p > -1 + \sqrt{1+n}$;

II. radiae $L = g\sqrt{\frac{e}{s}\left\{\frac{s}{e} - p - \sqrt{\frac{b^2}{4e}\left\{p^2 - 2p\frac{s}{e} + n\frac{s}{e}\right\}}\right\}}$

et $y = Y_2 = \sqrt{-\left\{L^2 - g^2\right\}\frac{b}{2} - g^2\sqrt{e \cdot f(p,n)}} = g\sqrt{\frac{e}{s}\left\{\frac{b}{2}p + \sqrt{e\left\{p^2 - 2p\frac{s}{e} + n\frac{s}{e}\right\}}\right\}}$

1) si est $p+$, $n+$, $p>\frac{s}{e}-\sqrt{\frac{b^2}{4e}\left(\frac{s}{e}-n\right)}$, $\frac{s}{e}>n$, $n>\frac{4s}{b^2}$, $p<\frac{s}{e}-\sqrt{\frac{s}{e}\left(\frac{s}{e}-n\right)}$

2) si est $p-$, $n-$, $p>1-\sqrt{1-n}$, $n<1$; $p<-\frac{s}{e}+\sqrt{\frac{b^2}{4e}\left(\frac{s}{e}+n\right)}$;

3) si est $p-$, $n+$ $\left\{\begin{array}{l} p<-\frac{s}{e}=\sqrt{\frac{b^2}{4e}\left(\frac{s}{e}-n\right)},\ n<\frac{s}{e} \text{ simulque } n<\frac{4s}{b^2}. \\ p<1+\sqrt{1+n} \end{array}\right.$

III. 1. Radix $K=L$ realis est $=g\sqrt{\frac{e}{s}\left(\frac{s}{e}-p\right)}$ | si est $p+$, $n+$,

$Y_1=Y_2=\sqrt{\frac{b}{2}\left(K^2-g^2\right)}=g\sqrt{\frac{e}{s}\left(\frac{b}{2}\cdot p\right)}$ | $p=\frac{s}{e}-\sqrt{\frac{s}{e}\left(\frac{s}{e}-n\right)}$, $n<\frac{s}{e}$;

2) $K=L=g$ est, $Y_1=Y_2=o$ est $\}$ si est $p=n=o$;

3) $K=g\sqrt{(1+p)}$, $Y_1=o$ $\}$ si est $p+$, $n+$, $p=-1+\sqrt{1+n}$;

4) $L=g\sqrt{(1-p)}$ est, si est $p-$, $n-$, $p=1-\sqrt{1-n}$, $n<1$;
$Y_2=o$

5) $K=L=o$ est, si est $p+$, $n+$, $p=n=\frac{s}{e}$;

$Y_1=Y_2=g\sqrt{\frac{b}{2}}$

6) $K=Y_2=o$ est, si est $p-$, $n-$, $p=n=1$.

§. 17. Quantitates reales simul sunt

I. radix L (aequ. 23.) $=g\sqrt{\frac{e}{s}\left\{\frac{s}{e}-p-\sqrt{\frac{b^2}{4e}\left(p^2-2p\frac{s}{e}+\frac{s}{e}n\right)}\right\}}$

et $y=Y_1=\sqrt{\frac{b}{2}\left(L^2-g^2\right)+g^2\sqrt{e\cdot f(p,n)}}=g\sqrt{\frac{e}{s}\left\{-\frac{b}{2}p-\sqrt{e\left(p^2-2p\frac{s}{e}+\frac{s}{e}n\right)}\right\}}$

si est $p-$, $n-$, $p<-\frac{s}{e}+\sqrt{\frac{b^2}{4e}\left(\frac{s}{e}+n\right)}$, $p>-\frac{s}{e}+\sqrt{\frac{s}{e}\left(\frac{s}{e}+n\right)}$

II. radix $K=g\sqrt{\frac{e}{s}\left(\frac{s}{e}-p\right)+\sqrt{\frac{b^2}{4e}\left(p^2-2p\frac{s}{e}+\frac{s}{e}n\right)}}$

et $y=Y_2=\sqrt{\frac{b}{2}\left(K^2-g^2\right)-g^2\sqrt{e\cdot f(p,n)}}=g\sqrt{\frac{e}{s}\left\{-\frac{b}{2}p+\sqrt{e\left(p^2-2p\frac{s}{e}+n\frac{s}{e}\right)}\right\}}$

1) si est $p+, n+, p < -1 + \sqrt{1+n}$;
2) si est $p-, n+$;
3) si est $p-, n-, p > -\frac{s}{e} + \sqrt{\frac{s}{e}\left(\frac{s}{e}+n\right)}$;

III. 1) Radix $K = L = g\sqrt{\frac{e}{s}\left(\frac{s}{e}+p\right)}$ est, si est $p-, n-, p = -\frac{s}{e} + \sqrt{\frac{s}{e}\left(\frac{s}{e}+n\right)}$

$$Y_1 = Y_2 = g\sqrt{\frac{e}{s}\cdot\frac{b}{2}\cdot p}$$

2) $K = L = g$ est, si est $p = n = o$;
$Y_1 = Y_2 = o$
3) $L = g\sqrt{(1-p)}$ est, si est $p-, n-, p = 1 - \sqrt{1+n}$;
$Y_1 = o$
4) $K = g\sqrt{(1+p)}$ est, si est $p+, n+, p = -1 + \sqrt{1+n}$;
$Y_2 = o$
5) $L = o = Y_1$ est, si est $p-, n-, p = n = 1$.

§. 18. Si $x = o$ ponitur, fit

in aequ. 5) $$y = Y_1 = g\sqrt{\frac{b}{2} + \sqrt{\left(\frac{b^2}{4} - e - en\right)}}$$

$$y = Y_2 = g\sqrt{\frac{b}{2} - \sqrt{\left(\frac{b^2}{4} - e - en\right)}}$$

in aequ. 6) $$y = Y_1 = g\sqrt{-\frac{b}{2} + \sqrt{\left(\frac{b^2}{4} - e - en\right)}}$$

$$y = Y_2 = g\sqrt{-\frac{b}{2} - \sqrt{\left(\frac{b^2}{4} - e - en\right)}}$$

Est igitur I. in aequ. 6) $Y_1 = o$, Y_2 imaginarium
II. in aequ. 5) $Y_2 = o$, $Y_1 = g\sqrt{b}$, } si est $e < \frac{b^2}{4}$, $n-, n = 1$;

III. in aequ. 6) Y_1 realis quantitas, Y_2 imag.
IV. in aequ. 5) Y_1 realis quantitas $> g\sqrt{b}$, Y_2 imag. } si est $e < \frac{b^2}{4}$, $n-, n > 1$;

V. in aequ. 5) $Y_1 < g\sqrt{b}$ sed $> g\sqrt{\frac{b}{2}}$, $Y_2 < g\sqrt{\frac{b}{2}}$, si est $e < \frac{b^2}{4}$, $\frac{b^2}{4} - e = s$, $\begin{cases} n-, n < 1 \\ n = o \\ n+, \frac{s}{e} > n \end{cases}$

VI. in aequ. 5) $Y_1 = Y_2 = g\sqrt{\frac{b}{2}}$, si est $e < \frac{b^2}{4}$, $\frac{b^2}{4} - e = s$, $n+$, $\frac{s}{e} = n$.

Cum, ut jam supra dictum est, tangentes curvarum, ubi $\frac{dy}{dx}=o$ est, axi abscissarum parallelae sint, apparet, curvas expressas aequu. 5, 6 tales tangentes habere in iis casibus, quibus $x=o$ est simulque $y=Y_1$ reale $Y_1{}^2>\frac{b}{2}g^2$; deinde in iis, quibus $\left\{x^2+\frac{d}{2e}+\frac{b}{2e}y^2\right\}=o$ est, $Y_1\left\{Y_1{}^2+\frac{b}{2}(x^2+g^2)\right\}$ reale, cf. §. 16, 17; denique in iis casibus, quibus $x=o$, Y_2 reale $<g\sqrt{\frac{b}{2}}$ est aut (§. 16. 17), $\left\{x^2+\frac{d}{2e}+\frac{b}{2e}y^2\right\}=o$ est, $Y_2\left\{Y_2{}^2+\frac{b}{2}\left\{x^2+g^2\right\}\right\}$ reale.

§. 19. $\frac{dy}{dx}=-\frac{ex}{y}\left\{\frac{x^2-g^2\cdot(1+p)+\frac{b}{2e}y^2}{y^2+\frac{b}{2}\left\{x^2-g^2\right\}}\right\}=\frac{0}{0}$ fit, si eodem tempore, quo $x=o$ est, $y=o$ aut $y^2-\frac{b}{2}g^2=o$ est, aut si polynomium $x^2-g^2\ (1+p)+\frac{b}{2e}\cdot Y^2=o$ positum efficit, ut $y=o$ aut $y^2+\frac{b}{2}(x^2-g^2)=o$ fiat.

Quantitatis incerti $\frac{dy}{dx}=\frac{0}{0}$ valor verus hoc modo invenitur.

Differentia aequationem $\frac{dy}{dx}=-\frac{ex}{y}\left\{\frac{x^2-g^2\cdot(1+p)+\frac{b}{2e}y^2}{y^2+\frac{b}{2}(x^2-g^2)}\right\}$, in qua, ut notum est, et dy et dx constantes quantitates ponuntur.

Tum invenies

$$24)\quad \frac{dy^2}{dx^2}\left\{3y^2+\frac{b}{2}\left\{x^2-g^2\right\}\right\}+2bxy\frac{dy}{dx}=-e\left\{3x^2-g^2\cdot(1+p)+\frac{b}{2e}y^2\right\}$$

ex qua aequatione deducuntur aequationes hae:

$$25)\quad \frac{dY_1}{dx}\sqrt{\frac{2e}{b}(1+p}\quad \text{si est } x=o,\ Y_1=o \text{ (cf. §. 18.)},\ b-,\ a+;$$

$$26)\quad \frac{dY_2}{dx}=\sqrt{-\frac{2e}{b}(1+p)}\quad \text{si est } x=o,\ Y_2=o \text{ (cf. §. 18.)},\ b+,\ a-;$$

$$27)\quad \frac{dY_1}{dx}=\sqrt{-\frac{1}{b}\left\{\frac{b^2}{4}-e-ep\right\}}\quad \text{si est } x=o,\ Y_1=Y_2=g\sqrt{\frac{b}{2}},\ b+,\ a- \text{(cf. §. 18)}.$$

28) $\frac{dY}{dx}=\frac{x}{y}\left\{-\frac{b}{2}\pm\sqrt{\left(\frac{b^2}{4}-e\right)}\right\}$, si est $y^2+\frac{b}{2}(x^2-g^2)=o, x^2-g^2(1+p)\frac{b}{2e}y^2=o;$

29) $\frac{dY}{dx}=\sqrt{-\frac{4\,ex^2}{b(x^2-g^2)}}$, si est $y=o, x^2-g^2(1+p)=o.$

§. 20. Cum $\frac{dy}{dx}$ duplicem valorem realem obtineat,

in aequ. 25) si est $p+, p=o, p-$ et <1;
in aequ. 26) si est $p-$ et >1;
in aequ. 27) si est $\frac{b^2}{4}>e$, $\frac{b^2}{4}-e=s$, $p+$ et $>\frac{s}{e}$;
in aequ. 29) si est in aequ. 5) §. 4. $p-$, $n-$, $k=l=g\sqrt{(1-p)}$;
in aequ. 6) §. 4. $p+$, $n+$, $k=l=g\sqrt{(1+p)}$;

in aequ. 28) denique, si est in aequ. 5, $e<\frac{b^2}{4}$, $\frac{b^2}{4}-e=s, p=\frac{s}{e}-\sqrt{\frac{s}{e}\left(\frac{s}{e}-n\right)}, \frac{s}{e}<n$;

$$K \text{ igitur } = L = g\sqrt{\frac{e}{s}\left(\frac{s}{e}-p\right)} \text{ cf. §. 16, III, 1;}$$

$$Y_1 = Y_2 = g\sqrt{\frac{b\cdot ep}{2s}}$$

si est in aequ. 6) $e<\frac{b^2}{4}$, $\frac{b^2}{4}-e=s$, $p=-\frac{s}{e}+\sqrt{\frac{s}{e}\left(\frac{s}{e}+n\right)}$;

$$K \text{ igitur } = L = g\sqrt{\frac{e}{s}\left(\frac{s}{e}+p\right)} \text{ cf. §. 17, III, 1;}$$

$$Y_1 = Y_2 = g\sqrt{\frac{b\cdot ep}{2s}}.$$

curvae in casibus enumeratis puncta duplicia habent, quorum tangentes trigonometricae radicibus quantitatis $\left(\frac{dy}{dx}\right)^2$ exprimuntur. Si angulum, quem altera tangens cum abscissarum axi format in aequ. 25. 26, 27, 29 per φ denotas, angulus alterius tangentis $=180-\varphi=\varphi'$ est.

In aequ. 25) fit $\frac{dY_1}{dx}=o$
In aequ. 26) fit $\frac{dY_2}{dx}=o$
} si est $p-$, $p=1$;

In aequ. 27) fit $\frac{dY_1}{dx}=o$, si est $\frac{b^2}{4}-e=s, p+, p=\frac{s}{e}$.

In aequ. 29) fit $\frac{dy}{dx}=\infty$, si est $x=\sqrt{g}$.

quamobrem curvae, si, substituto valore infinite parvo h in locum quantitatis x, y valorem realem accipit, curvae tangentes axi abscissarum parallelas habent, ubi $x=o$ est.

Contra habent curvae tangentes axi odinatarum parallelas, si in aequ. 29, $\frac{dy}{dx}=\infty$ est et substituto $x\pm h$ in locum x quantitas y valores reales obtinet.

§. 21. Si $e\{x^4-2g^2(1+p)x^2+g^4(1+n)\}=o$ fit in aequ. 5, 6, aut $\left\{\begin{matrix} k>o \\ k\gtrless g \end{matrix}\right\}$ substituto, aut $\left\{\begin{matrix} l>o \\ l\gtrless g \end{matrix}\right\}$, y igitur evanescit, sed $y^2+\frac{b}{2}\left(x^2-g^2\right)$ quantitas finita est, porro, si evanescente polynomio $\frac{b^2}{2}\left\{x^2-g^2\right\}-e\{x^4-2g^2(1+p)x^2+g^4(1+n)\}$, in aequ. 5, $g=\sqrt{-\frac{b}{2}\left(x^2-g^2\right)}$ in aequ. 6, $y=\sqrt{\frac{b}{2}\left(x^2-g^2\right)}$ quantitates reales sunt, simulque trinoium $e\left\{x^2-g^2(1+p)+\frac{b}{2e}y^2\right\}$ quantitas finita est, fit $\frac{dy}{dx}=\infty$, quamobrem tangentes ordinatarum axi parallelao sunt.

Cum $x=+\infty$ substituto $y=x\sqrt{-\frac{b}{2}\pm\sqrt{\frac{b^2}{4}-e}}$ fiat in aequ, 5,

$$y=x\sqrt{\frac{b}{2}\pm\sqrt{\frac{b^2}{4}-e}}$$ fiat in aequ. 6,

y igitur duos valores reales adoptet in hac, duos imaginarios in illa, si $\frac{b^2}{4}>e$ est, sequitur ut curvae aequatione 6 expressae, $x=+\infty$ substituto, tangentes tantum habeant. Quae tangentes, quia $\frac{dy}{dx}=\sqrt{\frac{b}{2}\pm\sqrt{\frac{b^2}{4}-e}}$ est, neque ordinatarum, neque abscissarum axi parallelae sunt.

Cap. III. De quotiente differentiali secundo.

§. 22. Est (§. 13.) $dy\left\{y^3+\frac{b}{2}x^2y+\frac{a}{2}y\right\}+\left\{ex^3+ex\frac{d}{2}+\frac{bx}{2}y^2\right\}dx=o$.

Quam aequationem si differentiaveris, invenies

$$30)\ \frac{d^2x}{dx^2}\left\{y^3+\frac{b}{2}yx^2+\frac{a}{2}y\right\}+\left(\frac{dy}{dx}\right)^2\left(3y^2+\frac{b}{2}x^2+\frac{a}{2}\right)+\frac{dy}{dx}\cdot 2bxy+e\left(3x^2+\frac{d}{2e}+\frac{b}{2e}y^2\right)=o$$

I. Si est $x^2+\frac{d}{2e}+\frac{b}{2e}y^2=o,\ y\left\{y^2+\frac{b}{2}x^2+\frac{a}{2}\right\}$ quantitas finita, $\frac{dy}{dx}=o$,

fit $\frac{d^2y}{dx^2}=\frac{-2ex^2}{y\left\{y^2+\frac{b}{2}\left(x^2-g^2\right)\right\}}$ in eo casu, quo $b+$, $a-$

fit $\frac{d^2y}{dx^2}=-\frac{-2ex^2}{y\left\{y^2-\frac{b}{2}\left(x^2-g^2\right)\right\}}$ in eo, quo $b-$, $a+$.

Uterque numerus fractus negativus est, si est $y=Y_1$,
positivus est, si est $y=Y_2$.

II. Si $x=o$, $y\left\{y^2\pm\frac{b}{2}\left(x^2-g^2\right)\right\}$ quantitas finita est, fit

$$\frac{d^2y}{dx^2}=-\frac{e\cdot\left\{\frac{b}{2e}y^2-g^2(1+p)\right\}}{y\left(y^2\mp\frac{bg^2}{2}\right)}=o,\ \text{ut sit } y=g\sqrt{\frac{2e(1+p)}{b}},$$

quo in casu etiam est $\frac{d^3y}{dx^3}=-\frac{6ex}{4y\left(y^2\mp\frac{bg^2}{2}\right)}=o$,

$\frac{d^4y}{dx^4}=-\frac{3e}{2y\left(y^2\mp\frac{bg^2}{2}\right)}$, quae fractio negativa quantitas est, si est $y=Y_1$,

positiva contra, si est $y=Y_2$.

Cum $x=o$ substituto in aequ. 6. fiat

$$y=Y_1=g\sqrt{-\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}-e-en\right)}}$$

in aeqn. 5) $y=Y_1=g\sqrt{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}-e-en\right)}}$

$$y=Y_1=g\sqrt{\frac{b}{2}-\sqrt{\left(\frac{b^2}{4}-e-en\right)}}$$

cum porro, si $b-$, $a+$ ponitur, $\frac{b}{2e}Y_1{}^2-g^2(1+p)=-g^2\left\{1+p-\frac{b^2}{4e}+\frac{b}{2e}\sqrt{\frac{b^2}{4}-e-en}\right\}$

si $b+$, $a-$ ponitur, $\frac{b}{2e}Y_1{}^2-g^2(1+p)=-g^2\left\{1+p-\frac{b^2}{4e}-\frac{b}{2e}\sqrt{\frac{b^2}{4}-e-en}\right\}$

si $b+$, $a-$ ponitur, $\frac{b}{2e}Y_2{}^2-g^2(1+p)=-g^2\left\{1+p-\frac{b^2}{4e}+\frac{b}{2e}\sqrt{\frac{b^2}{4}-e-en}\right\}$

fiat, sequitur, ut $y=Y_1$ valorem realem adoptet, $\frac{d^2y}{dx^2}$ negativum, ubi est $x=o$, si est in aequ. 6, §. 4,

$$p-, n-, n>1, p>-\frac{s}{e}+\sqrt{\frac{b^2}{4e}\left(n+\frac{s}{e}\right)}, e<\frac{b^2}{4}$$

in aequ. 5. §.4. 1) $p+, n-, p<\frac{s}{e}+\sqrt{\frac{b^2}{4e}\left(\frac{s}{e}+n\right)}, e<\frac{b^2}{4}$

2) $p+, n+, p<\frac{s}{e}+\sqrt{\frac{b^2}{4e}\left(\frac{s}{e}-n\right)}, e<\frac{b^2}{4}, \frac{s}{e}>n$

3) $p-, n-, e<\frac{b^2}{4}$

4) $p-, n+, \frac{s}{e}>n, e<\frac{b^2}{4}$.

ut Y_2 valorem realem obtineat, $\frac{d^2y}{dx^2}$ negativum, ubi est $x=o$, si est in aequ. 5, §. 4.

1) $p-, n+, e<\frac{b^2}{4}, p<-\frac{s}{e}+\sqrt{\frac{b^2}{4e}\left(\frac{s}{e}-n\right)}, n<\frac{s}{e}$

2) $p+, n+, e<\frac{b^2}{4}, p>\frac{s}{e}-\sqrt{\frac{b^2}{4e}\left(\frac{s}{e}-n\right)}, n<\frac{s}{e}$

3) $p+, n-, e<\frac{b^2}{4}, p>\frac{s}{e}-\sqrt{\frac{b^2}{4e}\left(\frac{s}{e}+n\right)}, n<1$

4) $p-, n-, e<\frac{b^2}{4}, p<-\frac{s}{e}+\sqrt{\frac{b^2}{4e}\left(\frac{s}{e}+n\right)}$

Superfluum esse videtur etiam illos quantitatum p, n valores enumerare, qui substituti, ut $\frac{d^2y}{dx^2}$ positiva quantitas fiat, efficiunt, quia jam notum est, ubi $\frac{d^2y}{dx^2}=-\frac{e\left\{\frac{b}{2e}y^2-g^2(1+p)\right\}}{y\left\{y^2\mp\frac{b}{2}g^2\right\}}$ negativa quantitas sit, curvae igitur maxima habeant et abscissarum axi concavum latus praebeant.

§. 23. Si $\frac{dy}{dx}=\frac{0}{0}$ est, fit etiam $\frac{d^2y}{dx^2}=\frac{0}{0}$ et incertum est, utrum curvae axi abscissarum concavum latus offerant, an convexum, an denique puncta inflexionis habeant, quia $\frac{d^2y}{dx^2}=\frac{0}{0}$ aut positivum, aut

negativum, aut $=o$, aut $=\pm\infty$ esse potest. Numeri fracti incerti $\frac{d^2y}{dx^2}=\frac{0}{0}$ verus valor inveniendus est, id quod hoc modo fit. Si in locum quantitatis x substitueris $x\pm h$, sumes $x=\omega$ (quo numero ω substituto et numerator fracti numeri et denominator evanescit) h autem infinite parvum. Quibus factis valorem inventum quantitatis $y=\varphi(\omega+h)$ in seriem infinitam commutabis, pro x in quotiente differentiali secundo pones $(x+h)$, pro y seriem inventam et denique aequalibus factoribus numeratoris et denominatoris emotis quantitati h infinite parvae valorem zero tribues. Si omnia modo commemorata feceris, invenies

$$31)\quad \frac{d^2y}{dx^2}=-\frac{\left\{\frac{b}{2}+\sqrt{\frac{b^2}{4}-e}\right\}\left\{\frac{b}{2}+\sqrt{\frac{b^2}{4}-e+en}\right\}}{g\frac{b}{2}\left\{-1+\sqrt{\frac{e}{s}\left(\frac{s}{e}+n\right)}\right\}\sqrt{\frac{b}{2}\left\{-1+\sqrt{\frac{e}{s}\left(\frac{s}{e}+n\right)}\right\}}}\text{, si est } b-,\ a+,$$

$$\omega^2=g^2\frac{e}{s}\sqrt{\frac{s}{e}\left(\frac{s}{e}+n\right)}\ \text{cf. §. 9. II, } \gamma,\ e<\frac{b^2}{4},\ y=Y_1$$

$$32)\quad \frac{d^2y}{dx^2}=-\frac{\left\{\frac{b}{2}-\sqrt{\frac{b^2}{4}-e}\right\}\left\{\frac{b^2}{4}-\sqrt{\frac{b^2}{4}-e-en}\right\}}{g\frac{b}{2}\left\{1-\sqrt{\frac{e}{s}\left(\frac{s}{e}-n\right)}\right\}\sqrt{\frac{b}{2}\left\{1-\sqrt{\frac{e}{s}\left(\frac{s}{e}-n\right)}\right\}}}\text{, si est } b+,\ a-,\ \frac{b^2}{4}>e$$

$$\text{et §. 9. II, } \alpha,\ \omega^2=g^2\frac{e}{s}\sqrt{\frac{s}{e}\left(\frac{s}{e}-n\right)},\ y-Y_1.$$

Ergo curvae aequ. 6. si est $\frac{b^2}{4}>e$, $p-$, $n-$, $n\gtrless 1$,

curvae aequ. 5. si est $\frac{b^2}{4}>e$, $p+$, $n+$,

ubi est $f=h=x$ §. 9. II, γ, α, nodos habent non puncta inflexionis.

Si in aequ. 6. $\frac{b^2}{4}>e, p-, n-, n=1$ est, fit alter valor quantitatis $y=Y_2=x\sqrt{\frac{b}{2}-\sqrt{\frac{b^2}{4}-e}}$,

quamobrem puncta inflexionis existere non possunt, quamquam valor inventus quantitatis $\frac{d^2y}{dx^2}=o$ fit.

Si $\left\{\begin{matrix} b-,\ a+ \\ b+,\ a- \end{matrix}\right\}$ ponitur, $\omega=f=h=g$, §. 9. β, fit $\frac{d^2y}{dx^2}=-\infty$.

Quia autem y valores reales tum tantum adoptat, si $x=(\omega-h)$ in aequ. 5, $x=(\omega+h)$ in aequ. 6. substituitur, curvis inflexionis puncta non sunt, sed reversionis.

Si, substituto $b+$, $a-$, $\omega=f=h=o$, $\frac{b^2}{4}>e$, $p+$, $n+$, $p=n=\frac{s}{e}$,

fit $\frac{d^2y}{dx^2}=-\frac{\left\{\frac{b}{2}+\sqrt{\frac{b^2}{4}-e}\right\}}{g\sqrt{\frac{b}{2}}}$ curvae, quia (§. 20. aequ. 27.) $\frac{dY_1}{dx}=o$ est, maxima habent, ubi

$x=o$ est, $y=g\sqrt{\frac{b}{2}}$.

Si $y=o$, $x^2=g^2(1+p)$, $k=l$ est, invenies, prout $a+$, $b-$ aut $b-$, $a+$ est

$$33)\quad \frac{d^2y}{dx^2}=-\frac{e(\omega^2+g^2)}{(\omega^2-g^2)\sqrt{\frac{eb}{2}\left\{g^2-\omega^2\right\}}},$$

$$34)\quad \frac{d^2y}{dx^2}=-\frac{e(\omega^2+g^2)}{(\omega^2-g^2)\sqrt{\frac{eb}{2}\left\{\omega^2-g^2\right\}}},$$

e quibus formulis concludendum est, curvas expressas

aequ. 5, nodos habere, si est $p-$, $n-$, $p=1-\sqrt{1-n}$, $n<1$, $\omega=k=g\sqrt{(1-p)}$ cf. §. 7. II, γ,

aequ. 6, nodos habere, si est $p+$, $n+$, $p=-1+\sqrt{1+n}$, $\omega=k=g\sqrt{(1+p)}$ cf. §. 7. II, α,

Cum si §. 7. III, $p-$, $n-$, $p=n=1$, $\omega=k=o$ est.

$$35)\quad \frac{d^2y}{dx^2}\text{ igitur}=\sqrt{\frac{2e}{bg2}},$$ $\frac{dy}{dx}$ autem $=o$ sit §. 20. aequ. 26. curvae aequ. 5. nodos non habent, sed minima, ubi $x=o$ est.

Si posueris, ut sit $d=-2g^2e(1+p)$ positiva quantitas, p igitur negativa quantitas majorque unitate, $c=g^4(1+n)e=o$, n igitur negativum et $=1$, $a-$, $b+$, $x=\pm h$, $y=\sqrt{\frac{a}{2}-\sqrt{\frac{a^2}{4}}}=o$, substituto $x=o$, invenies

$$36)\quad \frac{d^2y}{dx^2}=\frac{2eg^2(p-1)\left\{-3g^2eb-\frac{12g^2}{b}(p-1)^2e^2-6bg^2e(p-1)\right\}+h^3\cdot\left\{\quad\right\}}{-g^3\sqrt{\frac{8g^6\cdot b^3e^3(p-1)^3}{h^2}}\left\{1+\frac{h^2}{2g^8b^3e(p-1)}\left(\quad\right)\right\}^{\frac{3}{2}}}$$

qui numerus fractus, quantitate h evanescente, $=o$ fit.

Nobis igitur concludendum eat, curvas aequ. 5, ubi $x=o$, $y=o$ est, inflexionis puncta habere.

Idem valet, ubi $x=o$, $y=o$ est in aequ. 6, si est $n=1$, $n-$, $d=-2g^2(1+p)e$, $p=o$, $p-$ et <1.

Si denique $a-$, $b+$, $c=e(1+n)g^4$ et $n+$, $n=o$, $n-<1$, $\frac{a^2}{4}=\frac{b^2}{4}\, g^4 = (1+n)eg^4$

$d=-2g^2e\,(1+p) > \frac{g^2b^2}{2}$ ponitur, fit, ubi $x=o$, $y=g\sqrt{\frac{b}{2}}$ est,

$$37)\quad \frac{d^2y}{dx^2} = \frac{g^5\sqrt{\left\{2e(1+p)-\frac{b^2}{2}\right\}^3}\left\{-\frac{1}{8}b^2-\frac{1}{2}e(1+p)\right\}+hg^2\left\{2e(1+p)-\frac{b^2}{2}\right\}\left\{\quad\right\}}{\left\{\left\{2e(1+p)-\frac{b^2}{2}\right\}g^4\frac{b}{2}+h\left(\quad\right)\right\}^{\frac{3}{2}}}$$

et h evanescente

$$38)\quad \frac{d^2y}{dx^2} = -\frac{\left\{\frac{b^2}{4}+e(1+p)\right\}\sqrt{2}}{gb\sqrt{b}};$$

qua de causa curvae aequ. 5. nodos habent.

§. 24) Aequatio 30 transit, si ponitur $b-$, $a+$, $x=\pm\infty$, $y=\pm\infty$, in aequ.

$$39)\quad \frac{d^2y}{dx^2} = \frac{-\left\{x^2-\frac{b}{2}y^2\right\}^2x^2\left\{3y^2-\frac{b}{2}x^2\right\}+\left\{3ex^2-\frac{b}{2}y^2\right\}\left\{y^2-\frac{b}{2}x^2\right\}^2y^2+2bx^2y^2\left\{ex^2-\frac{b}{2}y^2\right\}\left\{y^2-\frac{b}{2}x^2\right\}}{y^3\left\{y^3-\frac{b}{2}x^2\right\}^3}$$

aut, si $y^2=mx^2$ sumitur, in aequ. hanc:

$$40)\quad \frac{d^2y}{dx^2} = -\frac{\left\{e-\frac{b}{2}m\right\}^2\left\{3m-\frac{b}{2}\right\}+\left\{3e-\frac{b}{2}m\right\}\left\{m-\frac{b}{2}\right\}m+2bm\left\{e-\frac{b}{2}m\right\}\left\{m-\frac{b}{2}\right\}}{\left\{m-\frac{b}{2}\right\}^3\sqrt{m^3}}$$

Quum difficile cognosci possit ex aequ. commemorata, utrum curvae, ubi $x=\pm\infty$, $y=\pm\infty$ est, axi abcissarum convexum latus praebeant, an concavum, necesse est, ut aequationes alias formemus, quarum auxilio valorem et signum quantitatis $\frac{d^2y}{dx^2}$ sine difficultate invenire possimus in omnibus casibus, quibus quantitati x valor permagnus tribuitur. Aequationes, quae nobis opus sunt facile invenientur, valore quantitatis

$$41)\quad y^2 = +\frac{b}{2}\left\{x^2-g^2\right\}+\sqrt{s\left\{x^4-2g^2x^2\frac{e}{s}\left\{\frac{s}{e}-p\right\}+g^4\frac{e}{s}\left\{\frac{s}{e}-n\right\}\right.}$$ cf. §. 4, in seriem

infinitam commutato, cujus in terminis exponentes quantitatis x semper minores fiant, ita ut sine errore maxima terminorum pars in eo casu, quo quantitati x permagnus quidem, sed non infinite magnus valor tribuitur, negligi possit. Differentiales secundi quotientes ex aequationibus auxilio serierum inventis

derivati transibunt, si quantitati x valorem permagnum dederis, in aequationes eas, quae nobis adhibendae sunt.

§. 25. Quantitas radicalis $\sqrt{s\left\{x^4-\frac{2g^2x^2e}{s}\left(\frac{s}{e}-p\right)+\frac{g^4e}{s}\left(\frac{s}{e}-n\right)\right\}}$

aut $x^2\sqrt{s}\left\{1-\frac{2g^2e}{x^2s}\left(\frac{s}{e}-p\right)+\frac{g^4e}{x^4s}\left(\frac{s}{e}-n\right)\right\}^{\frac{1}{2}}$ aequalis est seriei

42) huic: $\sqrt{s}\left\{x^2-g\frac{2e}{s}\left(\frac{s}{e}-p\right)-\frac{g^4e^2}{2x^2s^2}\cdot f(p,n)-\frac{g^6e^3}{2x^4s^3}\left(\frac{s}{e}-p\right)f(p,n)-\omega(x)\right\}$, qua in formula per $\omega(x)$ summa omnium terminorum, qui quolibet factorum horum: x^{-6}, x^{-8}, x^{-10} etc. multiplicati sunt, per $f(p,n)$ trinomium $\left(p^2-2p\frac{s}{e}+n\frac{s}{e}\right)$ designatur.

Serie 42. in aequ, 41. §. 24. substituta invenies:

$$43)\ y^2=y_1^2=x^2\left(\frac{b}{2}+\sqrt{s}\right)-g^2\left\{\frac{b}{2}+\frac{e}{s}\left(\frac{s}{e}-p\right)\sqrt{s}\right\}+\left\{\frac{g^4e^2}{2x^2s^2}f(p,n)+\frac{g^6e^3}{2x^4s^3}\left(\frac{s}{e}-p\right)f(p,n)+\omega(x)\right\}\sqrt{s}$$

$$44)\ y^2=y_2^2=x^2\left(\frac{b}{2}-\sqrt{s}\right)-g^2\left\{\frac{b}{2}-\frac{e}{s}\left(\frac{s}{e}-p\right)\sqrt{s}\right\}+\left\{\frac{g^4e^2}{2x^2s^2}f(p,n)+\frac{g^6e^3}{2x^4s^3}\left(\frac{s}{e}-p\right)f(p,n)+\omega x\right\}\sqrt{s}$$

Aequatio 43. dat:

$$45)\ y_1\frac{dy_1}{dx}=x\left(\frac{b}{2}+\sqrt{s}\right)+\left\{\frac{g^4e^2}{2x^3s^2}f(p,n)+\frac{g^6e^3}{x^5s^3}\left(\frac{s}{e}-p\right)f(p,n)-\frac{1}{2}\omega'(x)\right\}\sqrt{s}$$

$$46)\ y_1\frac{d^2y_1}{dx^2}=-\left(\frac{dy_1}{dx}\right)^2+\left(\frac{b}{2}+\sqrt{s}\right)-\left\{\frac{3g^4e^2}{2x^4s^2}f(p,n)+\frac{5g^6e^3}{x^6s^3}\left(\frac{s}{e}-p\right)f(p,n)+\frac{\omega''(x)}{2}\right\}\sqrt{s}$$

aequatio 44. contra:

$$47)\ y_2\frac{dy_2}{dx}=x\left(\frac{b}{2}-\sqrt{s}\right)-\left\{\frac{g^4e^2}{2x^3s^2}\cdot f(p,n)+\frac{g^6e^3}{5x^5s^3}\left(\frac{s}{e}-p\right)f(p,n)-\frac{1}{2}\omega'(x)\right\}\sqrt{s}$$

$$48)\ y_2\frac{d^2y_2}{dx^2}=-\left(\frac{dy_2}{dx}\right)^2+\left(\frac{b}{2}-\sqrt{s}\right)+\left\{\frac{3g^4e^2}{2x^4s^2}\cdot f(p,n)+\frac{5g^6e^3}{x^6s^3}\left(\frac{s}{e}-p\right)f(p,n)+\frac{1}{2}\omega''(x)\right\}\sqrt{s}$$

Si quantitati x tales valores tribuuntur, ut termini, quibus factores x^{-2}, x^{-3}, x^{-4} etc. sunt, sine errore negligi possint, fit

$$49)\ y_1^2\ \text{(cf. aequ. 43.)}=x^2\left(\frac{b}{2}+\sqrt{s}\right)-g^2\left\{\frac{b}{2}+\frac{e}{s}\left(\frac{s}{e}-p\right)\sqrt{s}\right\}$$

$$50)\ \frac{dy_1}{dx}\text{(cf. aequ. 45)}=\frac{x\left(\frac{b}{2}+\sqrt{s}\right)}{\sqrt{x^2\left(\frac{b}{2}+\sqrt{s}\right)-g^2\left\{\frac{b}{2}+\frac{e}{s}\left(\frac{s}{e}-p\right)\sqrt{s}\right\}}}$$

51) $y_1^3 \cdot \frac{d^2y_1}{dx^2}$ (cf. aequ. 46.) $=-g^2\left\{\frac{b}{2}+\sqrt{s}\right\}\left\{\frac{b}{2}+\frac{e}{s}\left\{\frac{s}{e}-p\right\}\sqrt{s}\right\}$

si est $p-, p=o, p+p \gtrless \frac{s}{e}+\sqrt{\frac{b^2}{4e}\frac{s}{e}}$

52) $y_1^3 \cdot \frac{d^2y_1}{dx^2}=-\frac{3}{2}\frac{g^4}{x^2}\left\{\frac{b}{2}\sqrt{s}+s\right\}(1+n)$, si est $p+$, $p=\frac{s}{e}+\sqrt{\frac{b^2}{4e}\frac{s}{e}}$; $n+$, $n=o$, $n-$ et $\gtrless 1$.

53) $y_1^3 \cdot \frac{d^2y_1}{dx^2}=-\frac{3}{2}\frac{g^4}{x^2}\left\{\frac{b}{2}\sqrt{s}+s\right\}(1+n)=o$, si est $p+$, $n-$, $p=\frac{s}{e}+\sqrt{\frac{b^2}{4e}\frac{s}{e}}$, $n=1$.

54) y_2^2 (cf. aequ. 44.) $=x^2\left\{\frac{b}{2}-\sqrt{s}\right\}-g^2\left\{\frac{b}{2}-\frac{e}{s}\left\{\frac{s}{e}-p\right\}\sqrt{s}\right\}$.

55) $\frac{dy_2}{dx}$ (cf. aequ. 47.) $=\frac{x\left\{\frac{b}{2}-\sqrt{s}\right\}}{\sqrt{x^2\left\{\frac{b}{2}+\sqrt{s}\right\}-g^2\left\{\frac{b}{2}-\frac{e}{s}\left\{\frac{s}{e}-p\right\}\sqrt{s}\right\}}}$.

56) $y_2^3\frac{d^2y_2}{dx^2}$ (cf. aequ. 48.) $=-g^2\left\{\frac{b}{2}-\sqrt{s}\right\}\left\{\frac{b}{2}-\frac{e}{s}\left\{\frac{s}{e}-p\right\}\sqrt{s}\right\}$,

si est $p+$, $p=o$, $p-$, $p \lessgtr -\frac{s}{e}+\sqrt{\frac{b^2}{4e}\frac{s}{e}}$

57) $y_2^3\frac{d^2y_2}{dx^2}=-\frac{3g^4}{2x^2}\left\{s-\frac{b}{2}\sqrt{s}\right\}(1+n)$, si est $p-$, $n+$, $n=o$, $n-n\lessgtr 1$, $p=-\frac{s}{e}+\sqrt{\frac{b^2}{4e}\frac{s}{e}}$

58) $y_2^3\frac{d^2y_2}{dx^2}=-\frac{3g^4}{2x^2}\left\{s-\frac{b}{2}\sqrt{s}\right\}$, si est $p-$, $n-$, $n-1$, $p=-\frac{s}{e}+\sqrt{\frac{b^2}{4e}\frac{s}{e}}$.

§. 26. $\frac{d^2y_1}{dx^2}$ (cf. aequ. 51, 52.) est igitur

quantitas qositiva, et curvae axi abscissarum convexum latus praebent, si est 1) $p+$, $n\frac{\pm}{0}$ $p>\frac{s}{e}+\sqrt{\frac{b^2}{4e}\cdot\frac{s}{e}}$;

2) $p+$, $n-$, $p=\frac{s}{e}+\sqrt{\frac{b^2}{4e}\cdot\frac{s}{e}}$, $n>1$;

quantitas negativa, si est 1) $p+$, $n-$, $p=\frac{s}{e}+\sqrt{\frac{b^2}{4e}\cdot\frac{s}{e}}$, $n<1$;

2) $p+$, $n=o$
3) $p+$, $n+$ } $p=\frac{s}{e}+\sqrt{\frac{b^2}{4e}\cdot\frac{s}{e}}$;

4) $p+$, $n\frac{\pm}{0}$, $p<\frac{s}{e}+\sqrt{\frac{b^2}{4e}\cdot\frac{s}{e}}$;

quantitas negativa, si est 5) $p = o,\ n \frac{\pm}{0}$;

6) $p -,\ n \ \frac{\pm}{0}$.

$\frac{d^2 y_2}{dx^2}$ (cf. aequ. 56. 57.) est

positiva quantitas, si est 1) $p -,\ n \frac{\pm}{0}\ p > -\frac{s}{e} + \sqrt{\frac{s}{e} \cdot \frac{b^2}{4e}}$;

2) $p -,\ n +$
3) $p -,\ n = o$ $\Big\}\ p = -\frac{s}{e} + \sqrt{\frac{s}{e} \cdot \frac{b^2}{4e}}$;

4) $p -,\ n -,\ n < 1,\ p = -\frac{s}{e} + \sqrt{\frac{b^2}{4e} \cdot \frac{s}{e}}$;

negativa quantitatis, si est 1) $p +,\ n \ \frac{\pm}{0}$

2) $p = o,\ n \frac{\pm}{0}$;

3) $p -,\ n \frac{\pm}{0}\ p < -\frac{s}{e} + \sqrt{\frac{s}{e} \cdot \frac{b^2}{4e}}$;

4) $p -,\ n -,\ n > 1,\ p = -\frac{s}{e} + \sqrt{\frac{b^2}{4e} \cdot \frac{s}{e}}$.

§. 27. Cum aequ. 49. 54. etiam scribi possit

59) $$y^2 = \frac{b}{2}(x^2 - g^2) \pm \sqrt{s \left\{ x^2 - g^2 \frac{e}{s}\left(\frac{s}{e} - p\right)\right\}^2}$$

et aequ. $y^2 = \frac{b}{2}(x^2 - g^2) \pm \sqrt{s \left\{ x^4 - \frac{e}{s}\left(\frac{s}{e} - p\right) 2 g^2 x^2 + g^4 \frac{e}{s}\left(\frac{s}{e} - n\right)\right\}}$

prorsus eadem sit, ac aequatio

60) $$y^2 = \frac{b}{2}(x^2 - g^2) \pm \sqrt{s \left\{ x^2 - g^2 \frac{e}{s}\left(\frac{s}{e} - p\right)\right\}^2 \pm g^4\, P s}$$

$\Big\{$in qua aequ.; $\quad -\left\{ g^4 \frac{e^2}{s^2}\left(\frac{s}{e} - p\right)^2 - g^4 \frac{e}{s}\left(\frac{s}{e} - n\right)\right\}$ per $P g^4$ designatur$\Big\}$;

cum porro, si quantitati x valores permagni tribuuntur, polynomium $s\left\{x^2 - g^2 \frac{e}{s}\left(\frac{s}{e} - p\right)\right\}^2 \pm g^4 \cdot P s$ in polynomium $s\left\{x^2 - g^2 \frac{e}{s}\left(\frac{s}{e} - p\right)\right\}^2$ transeat, aequ. 60. igitur in aequ. 59. mutetur, sequitur, ut, quae de secundo quotiente differentiali §. 26. dicta sunt, in iis casibus etiam valeant, in quibus in aequ. 6. $\frac{b^2}{4} > e$ est et x valores permagnos accipit.

§. 28. Utrum curvae aequ. datae in aliquo puncto, ubi tangentes trigonometricae neque $=o$ sunt, neque duplicem valorem realem obtinent, puncta inflexionis habeant, nec ne, auxilio secundi quotientis differentialis cognosci non potest, quia aequationum theoria adjumenta non praebet aequationis hujus solvendae: $\frac{d^2y}{dx^2}\left\{3y^2+\frac{b}{2}(x^2-g^2)\right\}+2bxy\frac{dy}{dx}+\left\{3ex^2-g^2\ e(1+p)+\frac{b}{2}y^2\right\}=o$, nisi quanitatibus b, e, p, n, valores speciales tribuis. — Quamquam igitur in plurimis casibus valores coordinatarum, quibus singula inflexionis puncta determinantur, inveniri non possunt, limites tamen, in quibus illa puncta inclusa sunt, indicari possunt. In curvis limitatis hoc fit, si illa puncta respicimus, ubi aut maxima sunt, aut minima, aut curvae revertuntur; in curvis, quae infinitis ramis porriguntur, si punctorum modo commemoratorum eorumque, qui ad valorem infinite magnum quantitatis x pertinent, quotientem secundum differentialem accuratius disquirimus.

Quae disquisitio cum in singulis jam curvarum generibus versetur, necesse est, ut dicatur, sub quibus conditionibus curvae aequationis datae limitatae fiant, sub quibus ramos in infinitum se porrigentes habeant, deinde ut singula curvarum genera componantur, quae maxima habeant, quae minima, quae maxima et minima, quae neque maxima, neque minima. Quum vero ad puncta inflexionis determinanda necesse sit, ut scias, quot reversionis puncta curvis sint, quia respiciendum est, utrum curvae in punctorum reversionis propinquitate abscissarum axi convexum latus offerant, an concavum, etiam in disponendis curvarum generibus puncta reversionis non negliges et disquires, qualis ibi sit quotiens differentialis secundus.

§. 29. Ex iis, quae de ordinatis jam dicta sunt, sequitur, ut curvae aequ. 5. expressae certis finibus conclusae sint, curvae aequ. 6. expressae ramos infinitos habeant.

Si $y=o$ est aut $y\pm\frac{b}{2}(x^2-g^2)=o, \frac{dy}{dx}=\infty, \frac{d^2y}{dx^2}=\infty$, curvis igitur puncta inflexionis sunt, incertum est, utrum, sive quantitas x crescit sive decrescit, curvae abscissarum axi concavum latus praebeant, an convexum. Ut cognoscamus, utrum fiat, necesse est, ut in locum valoris x, quo substituto aut y evanescit, aut $\left\{y^2\mp\frac{b}{2}(x^2-y^2)\right\}$, ponatur $(x\pm h)$ et h tam parvum supponatur, ut omnes dignitates quantitatis h, quarum exponentes majores sunt, quam unitas, negligere tibi liceat.

Suustituto $(x+h)$ in locum $x=k$ aut $x=l$ transit aequ. 5. §. 4.

in aequ 61. $$y^2=\frac{-4ehx\{x^2-g^2(1+p)\}}{b(x^2-g^2)}$$

aequ. 6, in aequ. 62) $$y^2=\frac{4ehx\{y^2-g^2(1+p)\}}{b(x^2-g^2)}$$

$\frac{d^2y}{dx^2}$ ex aequ. 5. derivatum in 63) $$\frac{d^2y}{dx^2}=-\sqrt{\frac{x\{x^2-g^2(1+p)\}}{-4ebh^3(x^2-g^2)}}$$

ex aequ. 6. derivatum in 64) $$\frac{d^2y}{dx^2}=-\sqrt{\frac{x\{x^2-g^2(1+p)\}}{4ebh^3(x^2-g^2}}$$

Si ponitur $(x+h)$ pro $x=f$, $x=h$, fit in aequ 5.

65) $$y^2=-\frac{b}{2}(x^2-g^2)\pm\sqrt{hx\left\{b^2x^2-4ex^2-b^2g^2+4g^2e\cdot(1+p)\right\}}$$

66) $$\frac{d^2y}{dx^2}=\pm\frac{1}{8}\sqrt{\frac{2x\{b^2x^2-4ex^2-b^2g^2+4g^2e(1+p)\}}{h^3\,b(x^2-g^2)}}$$

in aequ. 6; 67) $$y^2=\frac{b}{2}(x^2-g^2)\pm\sqrt{hx\left\{x^2b^2-4ex^2-b^2g^2+4g^2e(1+p)\right\}}.$$

68) $$\frac{d^2y}{dx^2}=\mp\frac{1}{8}\sqrt{\frac{2x\{b^2x^2-4ex^2-b\,g^2+4g^2e(1+p)\}}{h^3\,b(x^2-g^2)}}$$

Ex aequu. 61, 63; 62, 64; concludimus, curvas in propinquitate eorum punctorum, quibus curvae axi abcissarum secantur simulque revertuntur, axi illi latus concavum offerre; ex aequ. 65, 66; 67, 68; curvas in vicinitate eorum punctorum, quae ad $x=f$, $x=h$ pertineant, concavum latus

praebere, si est $y^2=\mp\frac{b}{2}(x^2-g^2)+\sqrt{hx\left\{x^2b^2-4ex^2-b^2g^2+4g^2e(1+p)\right\}}$

convexum, si est $y^2=\mp\frac{b}{2}(x^2-g^2)-\sqrt{hx\left\{x^2b^2-4ex^2-b^2g^2+4g^2e(1+p)\right\}}$

Eorum auxilio, quae modo et jam supra de quotiente secundo differentiali dicta sunt, in omnibus casibus sine difficultate cognosci potest, utrum curvae inflexionis puncta habeant, nec ne.

Reliquum nunc est, ut disquiramus, utrum curvae, quae in infinitum se porrigunt, asymptotas habeant, nec ne, ubi asymptotis et curvis puncta communia sint, denique utrum ordinatae curvarum majores sint, an asymptotarum.

Cap. IV. De asymtotis.

§. 30. Summus terminus aequationis 6. est,

$$y^4-bx^2y^2+ex^4$$

quod trinomium, si $e<\frac{b^2}{4}$ est, in factores hos reales dissolvi potest:

$$y^2-x^2\left\{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}-e\right)}\right\}$$

$$y^2-x^2\left\{\frac{b}{2}-\sqrt{\left(\frac{b^2}{4}-e\right)}\right\}.$$

Curvae aequ. 6. igitur habent, si $e<\frac{b^2}{4}$ est, duas rectas lineas asymptotas, quarum aequationes sunt:

$$y^2 = x^2 \left\{\frac{b}{2} + \sqrt{\frac{b^2}{4} - e}\right\}$$

$$y^2 = x^2 \left\{\frac{b}{2} - \sqrt{\frac{b^2}{4} - e}\right\}$$

Asymptotis et curvis puncta communia sunt, si quolibet valore quantitatis x substituto ordinatae et asymptotarum et curvarum eosdem valores accipiunt. Puncta illa igitur, quae curvis et asymptotis communia sunt, invenies, si ordinatis et abscissis asymptotarum et curvarum iidem valores tribuuntur et hoc facto ex earum aequationibus modo x, modo y emovetur.

Cum necesse sit, ut $y^2 = \frac{b}{2}(x^2 - g^2) + \sqrt{s\left\{x^4 - 2g^2 x^2 \frac{e}{s}\left(\frac{s}{e} - p\right) + g^4 \frac{e}{s}\left(\frac{s}{e} - n\right)\right\}}$

transeat in $y^2 = x^2\left(\frac{b}{2} + \sqrt{s}\right)$, suppones, ut sit

$$s\left\{x^4 - 2g^2 x^2 \frac{e}{s}\left(\frac{s}{e} - p\right) + g^4 \frac{e}{s}\left(\frac{s}{e} - n\right)\right\} = s\,x^4 \pm bg^2 x^2 \sqrt{s} + \frac{b^2}{4} g^4$$

$$\text{aut } 2\,x^2 \cdot g^2 \left\{ep - s \mp \frac{b}{2}\sqrt{s}\right\} = g^4(1+n)e$$

$$\text{aut } x^2 = \frac{g^2}{2} \cdot \frac{e(1+n)}{pe - s \mp \frac{b}{2}\sqrt{s}}$$

Concludimus igitur, curvas aequ. 6. et asymptotam. cujus aequ. est: $y^2 = x^2\left(\frac{b}{2} + \sqrt{s}\right)$ puncta communia habere,

$$\text{si est } e < \frac{b^2}{4} \{n+;\ n=o;\ n-,\ n<1;\}\ p+,\ p > \frac{s}{e} + \sqrt{\frac{b^2}{4e}\frac{s}{e}}$$

$$n-,\ n>1;\ p+,\ p < \frac{s}{e} + \sqrt{\frac{b^2}{4e}\frac{s}{e}} \text{ aut } p-;$$

itemque curvas aequ. 6. et asymptotam, cujus aequatio est: $y^2 = x^2\left(\frac{b}{2} - \sqrt{s}\right)$,

$$\text{si est } e < \frac{b^2}{4},\ n+;\ n=o;\ n-,\ n<1;\ p+;\ p-,\ p < -\frac{s}{e} + \sqrt{\frac{b^2}{4e}\frac{s}{e}}$$

$$n-,\ n>1;\ p-,\ p > -\frac{s}{e} + \sqrt{\frac{b^2}{4e}\frac{s}{e}}.$$

Si est $n+;\ n=o;\ n-,\ n \lesseqgtr 1;\ p+,\ p = \frac{s}{e} + \sqrt{\frac{b^2}{4e}\frac{s}{e}}$

fit $x = \infty$, quamobrem curvae iis asymptotis secari non possunt, quae aequ, $y^2 = \left\{ \frac{b}{2} + \sqrt{s} \right\} x^2$ exprimuntur.

Idem valet de curvis et asymptotis, quarum aequatio

est $y^2 = x^2 \left\{ \frac{b}{2} - \sqrt{s} \right\}$, si est $n+$; $n = o$; $n-n \gtrless 1$; $p-$, $p = -\frac{s}{e} + \sqrt{\frac{b^2}{4e} \frac{s}{e}}$.

Si $n-$, $n = 1$; p$+$, $p \gtrless \frac{s}{e} + \sqrt{\frac{s}{e} \frac{b^2}{4e}}$ est, curvae et asymptotae, cujus aequ. $y^2 = x^2 \left\{ \frac{b}{2} + \sqrt{s} \right\}$ est, illud punctum commune habent, cujus coordinatae sunt $x = y = o$.

Eadem valent de curvis et asymptota, cujus aequ. $y^2 = x^2 \left\{ \frac{b}{2} - \sqrt{s} \right\}$ est,

si est $n-$, $n = 1$; $p - p \gtrless - \frac{s}{e} + \sqrt{\frac{b^2}{4e} \frac{s}{e}}$.

Si $n-$, $n = 1$; $p+$ $p = \frac{s}{e} + \sqrt{\frac{b^2}{4e} \frac{s}{e}}$ est, aequ. curvarum transit

in: $y^2 = \left\{ \frac{b}{2} + \sqrt{s} \right\} x^2$ aequationem asymptotae,

aut in $y^2 = \left\{ \frac{b}{2} - \sqrt{s} \right\} x^2 - g^2 b$.

Si $n-$, $n = 1$; $p-$, $p = -\frac{s}{e} + \sqrt{\frac{b^2}{4e} \frac{s}{e}}$ est, commutatur aequatio curvarum

in $y^2 = \left\{ \frac{b}{2} + \sqrt{s} \right\} x^2 - g^2 b$

aut in $y^2 = \left\{ \frac{b}{2} - \sqrt{s} \right\} x^2$ aequationem asymptotae alterius.

Quia, si aequationes asymptotarum cum aequu. 43. 44. comparaveris et ea, quae de punctis curvarum et asymptotarum communibus exposita sunt, respexeris, facillime cognosces, ubi curvarum ordinatae majores sint, quam asymptotarum ordinatae, ubi minores, hac de re jam nihil amplius dicendum est, finemque dissertationi impositurus sum iis in memoriam revocatis, quae sub finem capitis primi dicta sunt.

VITA.

Natus sum ego Joannes Mauritius Jakobi Prausnitiae a. d. IV. Nonas Martias a. MDCCCXII evangelicae fidei addictus patre Joan. Theophilo, tum regiorum vectigalium exactore, matre Beata Christiana e familia Esterliniana. Cum primum ad eam aetatem adolevi, qua pueri in scholam itare solent, me artes legendi scribendique docuerunt, qui ludi magistri erant Prausnitiae; in calculi principiis ipse me instituit pater. Quo mortuo me decem annos natum avunculus Carolus Christianus Esterlin in domum recepit, qui latinae linguae elementis me imbuendum Simoni, verbi divini Grottcoviae ministro, tradidit. Tribus annis praeterlapsis mater in gymnasium Vratislaviense ad aedem St. Mariae Magdalenae me duxit, cui tum Manso, vir doctissimus, praeerat, cujus in munus paulo post successit Kluge, vir erga me benevolentissimus. Septem annis ex illo tempore praeterlapsis, quo in quartum gymnasii ordinem receptus eram, et examine, quod abeuntium vocatur, ita superato, ut primae notae testimonium acciperem, civibus Universitatis litterarum Viadrinae Vratislaviensis a. d. XVI Calendas Majas a MDCCCXXXII ab Huschke tum rectore Magnifico adscriptus sum. *E Koecheri*, viri de me optime meriti, consilio litteris mathematicis et physicis plurimum per quadriennium temporis tribui, praesertim cum benevolentia professorum *Scholtz* et *de Boguslawski*, quibus propter singularem in me benevolentiam summas semper habebo gratias, regii observatorii amanuensis factus essem. In his litteris duces mihi erant DD. *Scholtz, Pohl, Koecher, Kletke*. In litteris philosophicis me instituerunt DD. *Rohowski, Braniss*, in litteris chemicis D. *Fischer*, in

historicis DD. *Stenzel*, *Wachler*, *Kutzen*, in philologicis denique DD. *Schneider*, *Ritschel*, *Rohowski*.

In his omnibus disciplinis quod tam eximios habui praeceptores semper admodum gaudebo nec, quantum iis debeam, unquam obliviscar.

Universitate Calendis Juliis a MDCCCXXXVI relicta et mense Februario a MDCCCXXXVII examine pro facultate docendi in gymnasiorum ordinibus superioribus e voto superato, a viro illustrissimo *Middeldorpf*, regio consiliario ecclesiastico, qui idem est regii paedagogici seminarii director, in numerum sodalium adscitus, in gymnasio ad aedem St. Mariae Magdalenae et in Fridericiano usque ad hoc tempus varias disciplinas docui. In excolenda facultate docendi utilissimis consiliis me benevolentissime adjuverunt V. V. illustrissimi Middeldorpf et professor Dr. *Schoenborn*, gymnasii ad aedem St. Mariae Magdalenae director, quibus gratissimo semper animo addictus ero.

SENTENTIAE CONTROVERSAE.

I. Sidera volantia sunt corpora planetis similia.

II. Observationes stellarum, quae lunares vocantur, aptissimae sunt ad longitudinem geographicam inveniendam.

III. In disquisitionibus mathematicis praestat ab universali ad speciale transire.

IV. Geometria maximam partem analysis applicata est.

V. In phaenomena terrestris magnetismi processu chemico effecta calore vim aliquam exerceri contendo.

Corrigenda.

P. 1 l. 15 lege: $-x_1 \sin\psi$ pro $= x^1 \sin\varphi$.

P. 2 l. 14 leg.: $-e\left(x^4 + \frac{d}{e}x^2 + \frac{c}{e}\right)$ pro $-e\left(x^2 + \frac{d}{e}x^2 + \frac{c}{e}\right)$

P. 5 l. 16 leg.: $\vartheta(x)$ pro $\vartheta(x) = o$

P. 6 l. 26 leg.: $p+, n-$ pro $p-, n-$

P. 9 l. 6 leg.: $p < \frac{s}{e}$ pro $p > \frac{s}{e}$

P. 15 l. 8 leg.: $\left(x^2 + \frac{d}{2e} + \frac{b}{2e}y^2\right)$ pro $y^2 + \frac{d}{2e} + \frac{b}{2e}y^2$

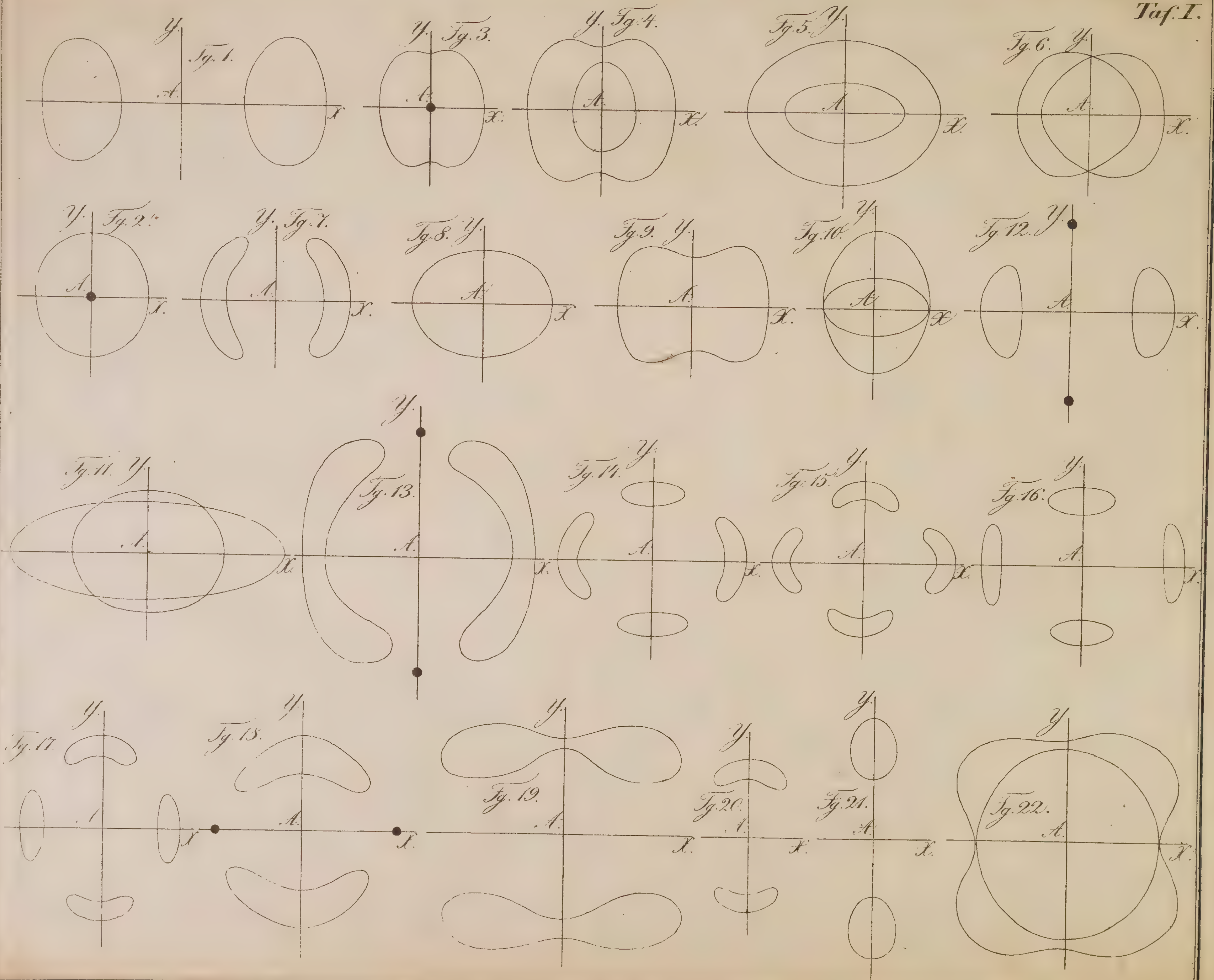
P. 20 l. 7 leg.: $x^2 - g^2\,(1+p) + \frac{b}{2e}y^2$ pro $x^2 - g^2\,(1+p) + \frac{b}{2e}Y_1{}^2$

P. 22 l. 9 leg.: trinomium pro trinoium

P. 28 l. 9 in aequ. 43. tribue tertio termino signum negativum

P. 29 l. 10 in aequ. 58. lege $p-, n-, n=1$ pro $p-, n-, n-1$

Taf. I.


Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.
Fig. 8.
Fig. 9.
Fig. 10.
Fig. 11.
Fig. 12.
Fig. 13.
Fig. 14.
Fig. 15.
Fig. 16.
Fig. 17.
Fig. 18.
Fig. 19.
Fig. 20.
Fig. 21.
Fig. 22.

Taf. II.

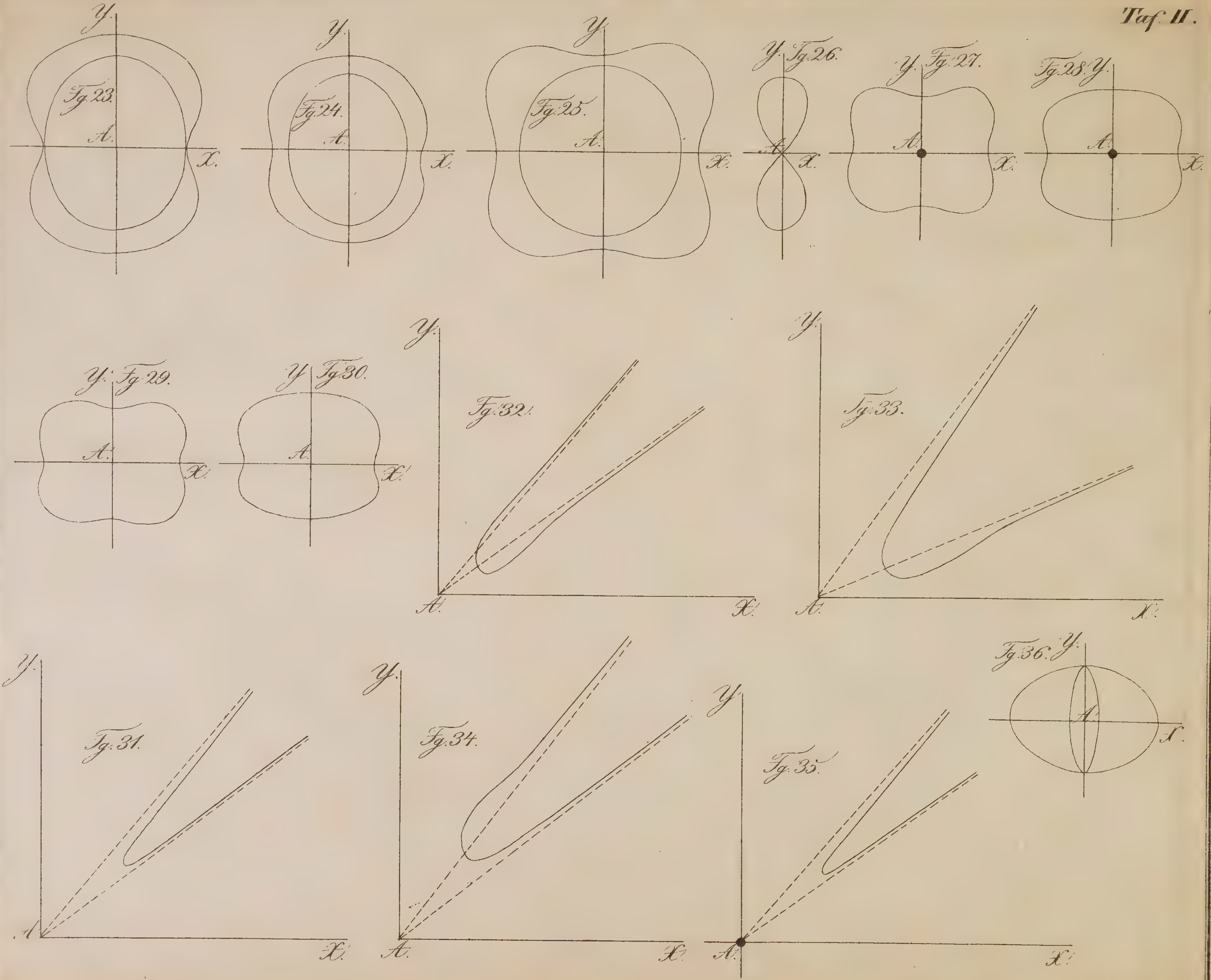
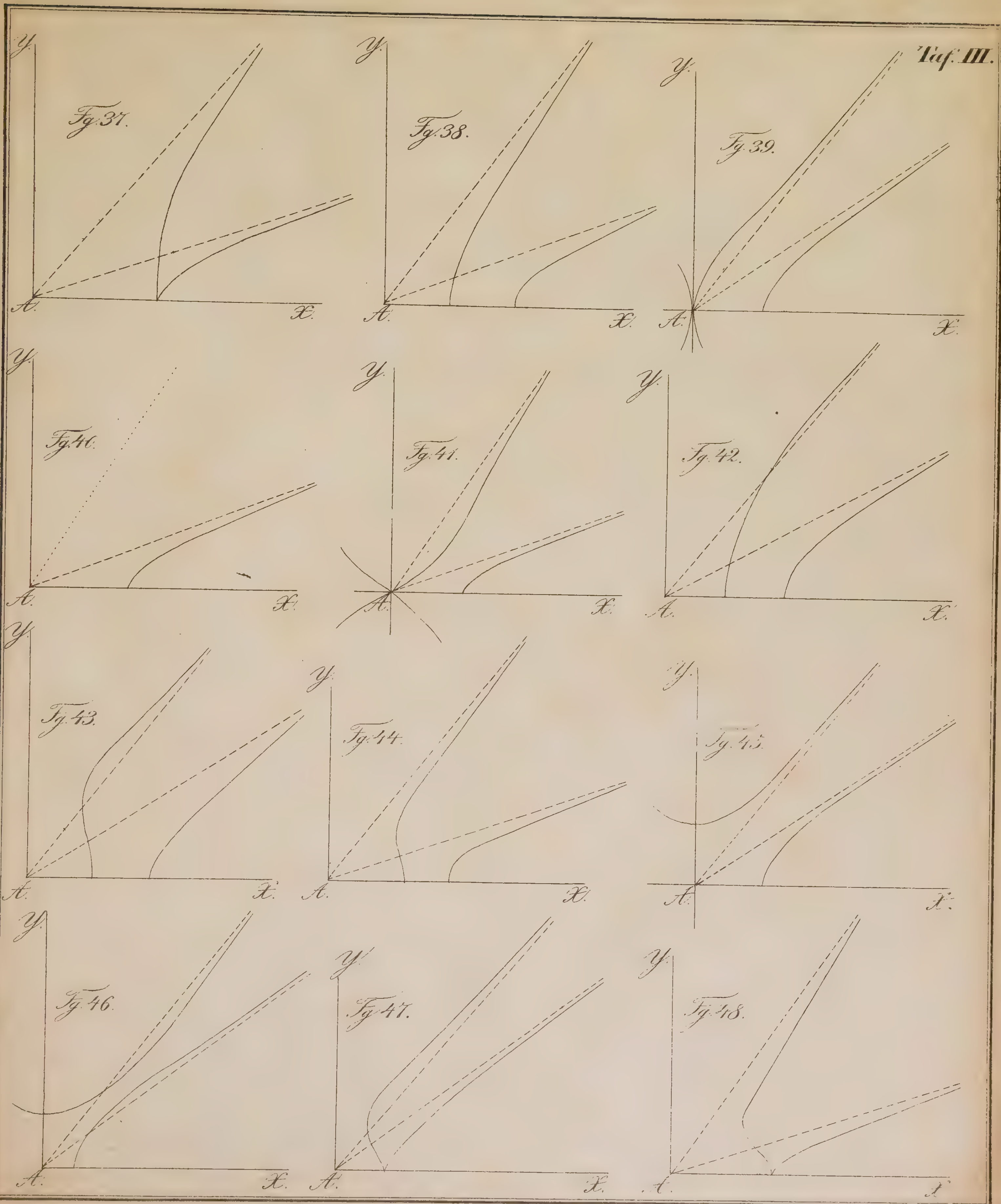


Fig. 23.
Fig. 24.
Fig. 25.
Fig. 26.
Fig. 27.
Fig. 28.
Fig. 29.
Fig. 30.
Fig. 31.
Fig. 32.
Fig. 33.
Fig. 34.
Fig. 35.
Fig. 36.

Taf. III.


Fig. 37.
Fig. 38.
Fig. 39.
Fig. 40.
Fig. 41.
Fig. 42.
Fig. 43.
Fig. 44.
Fig. 45.
Fig. 46.
Fig. 47.
Fig. 48.

Taf. IV.

